# MATHEMATICS BOOK

## ACCOUNTING PROFESSION OPTION

## for Rwandan Schools

Senior 6 Teacher Guide

Kigali, 2023

# FOREWORD

**Dear Teachers,**

Rwanda Basic Education Board is honoured to present the senior 6 teacher's guide for Mathematics in the Accounting Profession Option. This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics Subject. The Rwandan educational philosophy is to ensure that students achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities.

Specifically, the curriculum for Accounting Profession Option was reviewed to train quality Accountant Technicians who are qualified, confident and efficient for job opportunities and further studies in Higher Education in different programs under Accounting career advancement.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what students learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

High Quality Technician Accounting program is an important component of Finance and Economic development of the Rwanda Vision 2050, "***The Rwanda We Wan****t*" that aims at transforming the country's socioeconomic status. The qualified Technicians accountant will significantly play a major role in the mentioned socioeconomic transformation journey. Mathematics textbooks and teacher's guide were elaborated to provide the mathematical operations, algebraic functions and equations, and basic statistics that are necessary to train a Technician Accountant capable of successfully perform his/her duties.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum.

The Mathematics teacher's guide provides active teaching and learning techniques that engage students to develop competences. In view of this, your role as a Mathematics teacher is to:

- Plan your lessons and prepare appropriate teaching materials.

- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students works collaboratively with more knowledgeable and experienced people
- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group or individual work activities.
- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.
- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.
- Encourage individual, pair and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self-explanatory so that you can easily use it. It is divided in 3 parts:

The part I explains the structure of this book and gives you the methodological guidance;

The part II gives a sample lesson plan;

The part III details the teaching guidance for each concept given in the student book.

Even though this Teacher's guide contains the guidance on solutions for all activities given in the student's book, you are requested to work through each question before judging student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR Lecturers, Teachers from TTC and General Education and experts from different Education partners for their technical support. A word of gratitude goes also to the administration of Universities, Head Teachers and TTCs principals who availed their staffs for various activities.

**Dr. MBARUSHIMANA Nelson**

**Director General, REB.**

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I wish to express my appreciation to the people who played a major role in the development of this senior 6 teacher's guide for Mathematics in the Accounting profession option. It would not have been successful without active participation of different education stakeholders.

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I wish to extend my sincere gratitude to lecturers and teachers whose efforts during writing exercise of this teacher`s guide was very much valuable.

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**MURUNGI Joan**

**Head of Curriculum, Teaching and Learning Resources Department / REB**

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# PART I. GENERAL INTRODUCTION

## 1.1 The structure of the guide

The teacher's guide of Mathematics is composed of three parts:

The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the teacher on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit and these components are the same for all units. This part provides information and guidelines on how to facilitate students while working on learning activities. More other, all application activities from the textbook have answers in this part.

## 1.2 Methodological guidance

1.2.1 Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competence-based curriculum for pre-primary, primary, secondary education and recently the curriculum for profession options such as TTC, Associate Nurse and Accounting programs. This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering students' learning achievement and creating safe and supportive learning environment. It implies also that students have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Students develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes/values through learning activities.

In addition to the competences related to Mathematics, students also develop generic competences which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching. Generic competences are developed throughout all units of Mathematics as follows:

| Generic competences | Ways of developing generic competences |
|---|---|
| Critical thinking | All activities that require students to calculate, convert, interpret, analyse, compare and contrast, etc have a common factor of developing critical thinking into students |
| Creativity and innovation | All activities that require students to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of production/ finance/ economic have a common character of developing creativity into students |
| Research and problem solving | All activities that require students to make a research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into students. |
| Communication | During Mathematics class, all activities that require students to discuss either in groups or in the whole class, present findings, debate …have a common character of developing communication skills into students. |
| Co-operation, interpersonal relations and life skills | All activities that require students to work in pairs or in groups have character of developing cooperation and life skills among students. |
| Lifelong learning | All activities that are connected with research have a common character of developing into students a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling students to become life-long students who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support. |

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| Professional skills | Specific instructional activities and procedures that a teacher may use in the class room to facilitate, directly or indirectly, students to be engaged in learning activities. These include a range of teaching skills: the skill of questioning, reinforcement, probing, explaining, stimulus variation, introducing a lesson; illustrating with examples, using blackboard, silence and non-verbal cues, using audio – visual aids, recognizing attending behaviour and the skill of achieving closure. |

The generic competences help students deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

### 1.2.2 Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: *Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture*.

Some cross-cutting issues may seem specific to particular learning areas/ subjects but the teacher need to address all of them whenever an opportunity arises. In addition, students should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

| Cross-Cutting Issue | Ways of addressing cross-cutting issues |
|---|---|
| **Comprehensive Sexuality Education:** The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour. | Using different charts and their interpretation, Mathematics teacher should lead students to discuss the following situations: "Alcohol abuse and unwanted pregnancies" and advise students on how they can fight against them.<br><br>Some examples can be given when learning statistics, powers, logarithms and the related graphical interpretation. |
| **Environment and Sustainability:** Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Students need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability. | Using Real life models or students' experience, Mathematics Teachers should lead students to illustrate the situation of "population growth" and discuss its effects on the environment and sustainability. |

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| **Financial Education:**<br><br>The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life. | Through different examples and calculations on interest (simple and compound interests), interest rate problems, total revenue functions and total cost functions, supply and demand functions Mathematics Teachers can lead students to discuss how to make appropriate financial decisions. |
| **Gender:** At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc. | Mathematics Teachers should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process. |
| **Inclusive Education:** Inclusion is based on the right of all students to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity. | Firstly, Mathematics Teachers need to identify/recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where students can discuss how to cater for students with special educational needs. |

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| **Peace and Values Education:** Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society. | Through a given lesson, a teacher should:<br><br>Set a learning objective which is addressing positive attitudes and values,<br><br>Encourage students to develop the culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants;<br><br>Encourage students to respect ideas from others. |
| **Standardization Culture:** Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing. | With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people. |

### 1.2.3 Guidance on how to help students with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, need to be taught differently or need some accommodations to enhance the learning environment. This will be done depending to the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that students learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help students with special needs to stay on track during lesson and follow instruction easily;
- Vary the pace of teaching to meet the needs of each student. Some students process information and learn more slowly than others;
- Break down instructions into smaller, manageable tasks. Students with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a student who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both students will benefit from this strategy;
- Use multi-sensory strategies. As all students learn in different ways, it is important to make every lesson as multi-sensory as possible. Students with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.
- Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each student is unique with different needs and that should be handled differently.

**Strategy to help students with developmental impairment:**

- Use simple words and sentences when giving instructions;
- Use real objects that students can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The student should start with an activity that she/he can do already before moving on to something that is more difficult;
- Gradually give the student less help;
- Let the student with disability work in the same group with those without disability.

**Strategy to help students with visual impairment:**

- Help students to use their other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;
- Use tactile objects to help explain a concept;
- If the student has some sight, ask him/her what he/she can see;
- Make sure the student has a group of friends who are helpful and who allow him/her to be as independent as possible;
- Plan activities so that students work in pairs or groups whenever possible;

**Strategy to help students with hearing disabilities or communication difficulties**

- Always get the student's attention before you begin to speak;
- Encourage the student to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

**Strategies to help students with physical disabilities or mobility difficulties:**

- Adapt activities so that students who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g. the height of a table may need to be changed to make it easier for a student to reach it or fit their legs or wheelchair under;
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the student has one.

**Adaptation of assessment strategies:**

At the end of each unit, the teacher is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively. Therefore, the teacher is expected to do assessment that fits individual students.

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| **Remedial activities** | After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning.<br><br>These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students. |
| **Consolidation activities** | After introduction of any concept, a range number of activities can be provided to all students to enhance/ reinforce learning. |
| **Extended activities** | After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity. |

### 1.2.4. Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intends to improve students' learning and teacher's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

**Continuous/ formative assessment**

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, pair and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the student

- book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment
- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics teachers need to consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.
- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

**Summative assessment**

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of students and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies. There is also end of level/ cycle assessment in form of national examinations.

**When carrying out assessment?**

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know / can do, and to check whether the students are at the same level.
- During learning (formative/continuous): When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.

- After learning (summative): At the end of a section of work or a learning unit, the Mathematics Teacher has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

**Instruments used in assessment.**

- **Observation:** This is where the Mathematics teacher gathers information by watching students interacting, conversing, working, playing, etc. A teacher can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the teacher has to continue observing each and every activity.
- **Questioning**

a) Oral questioning: a process which requires a student to respond verbally to questions

b) Class activities/ exercise: tasks that are given during the learning/ teaching process

c) Short and informal questions usually asked during a lesson

d) Homework and assignments: tasks assigned to students by their teachers to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/instruments of assessment.

### 1.2.5. Teaching methods and techniques that promote active learning

The different learning styles for students can be catered for, if the teacher uses active learning whereby students are really engaged in the learning process.

The main teaching methods used in Mathematics are the following:

- **Dogmatic method** (the teacher tells the students what to do, What to observe, How to attempt, How to conclude)
- **Inductive-deductive method:** Inductive method is to move from specific examples to generalization and deductive method is to move from generalization to specific examples.
- **Analytic-synthetic method:** Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so

that it ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.

- **Skills lab method:** Skills lab method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.
- Problem solving method, Project method and Seminar Method.

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling
- Brainstorming
- Quiz Technique
- Discussion Technique
- Scenario building Technique

**What is Active learning?**

Active learning is a pedagogical approach that engages students in doing things and thinking about the things they are doing. Students play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, students are encouraged to bring their own experience and knowledge into the learning process.

| The role of the teacher in active learning | The role of students in active learning |
|---|---|
| – The teacher engages students through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities.<br>– He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.<br>– He provides supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.<br>– Teacher supports and facilitates the learning process by valuing students' contributions in the class activities. | A learner engaged in active learning:<br>– Communicates and shares relevant information with fellow students through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation);<br>– Actively participates and takes responsibility for his/her own learning;<br>– Develops knowledge and skills in active ways;<br>– Carries out research/investigation by consulting print/online documents and resourceful people, and presents their findings;<br>– Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking<br>– Draws conclusions based on the findings from the learning activities. |

**Main steps for a lesson in active learning approach**

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that students are involved in the learning process. Below are those main part and their small steps:

## 1. Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage students to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

## 2. Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of students' findings, exploitation, synthesis/summary and exercises/application activities.

◊ **Discovery activity**

**Step 1:**

- The teacher discusses convincingly with students to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to prompt / instigate collaborative learning, to discover knowledge to be learned)

**Step 2:**

- The teacher let students work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;
- He/she then monitors how the students are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

◊ **Presentation of students' findings/productions**

- In this part, the teacher invites representatives of groups to present their productions/findings.
- After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of students' productions.

◊ **Exploitation of students' findings/ productions**

- The teacher asks students to evaluate the productions: which ones are correct, incomplete or false
- Then the teacher judges the logic of the students' products, corrects those

which are false, completes those which are incomplete, and confirms those which are correct.

◊ **Institutionalization or harmonization (summary/conclusion/ and examples)**

- The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.

◊ **Application activities**

- Exercises of applying processes and products/objects related to learned unit/sub-unit
- Exercises in real life contexts
- Teacher guides students to make the connection of what they learnt to real life situations.
- At this level, the role of teacher is to monitor the fixation of process and product/object being learned.

**3. Assessment**

In this step the teacher asks some questions to assess achievement of instructional objective. During assessment activity, students work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow students to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

# PART II: SAMPLE LESSON

**Sample Lesson for unit 3: Integration**

**School Name:.....** **Teacher's name: .....**

| Term | Date | Subject | Class | Unit Nº | Lesson Nº | Duration | Class size |
|---|---|---|---|---|---|---|---|
| III | . . . . . /.../2023 | Mathematics | Senior 6 | 3 | ... ... of.... | 40 min | ... |
| **Type of Special Educational Needs to be catered for in this lesson and number of learners in each category** | | | | 3 slow students and 2 low vision students will be near their colleagues and work together | | | |

| | |
|---|---|
| **Unit title** | Integration |
| **Key unit competency:** | Determine the integration as the inverse of differentiation and apply it to solve problems that involves economics correctly |
| **Title of the lesson** | Basic integration formulas or immediate integration |
| **Instructional Objective** | Given a simple function, students will be able to accurately determine its integrations by considering the basic integration formulas . |
| **Plan for this Class (location: in / outside)** | The lesson is held indoors, the class is organized into groups ,3 slow students are scattered in different groups ,and 2 low vision students seat on the front desks near the blackboard in order to see and participate fully in all activities |
| **Learning Materials (for ALL studentss)** | Textbooks and charts containing the table of derivatives and integrations. |
| **References** | Mathematics senior six and Teacher's guide. |

<table>
<tr><th rowspan="3">Timing for each step</th><th colspan="2">Description of teaching and learning activity</th><th rowspan="3">Generic competences and cross cutting issues to be addressed + a short explanation</th></tr>
<tr><td colspan="2">Students work in pairs the introductory activity, and the correction is done on the chalk board by two students, one after another under the guidance of the teacher.<br><br>Then they discuss in groups the discovery activity, followed by the presentation by a sample group, interaction of students and harmonization of the results under the facilitation of the teacher. Next, they discuss in pairs the solved example and compare their results with the answer proposed in the book. Finally, the students are assigned individual tasks, and the correction is done on the chalk board, and the teacher winds up the lesson.</td></tr>
<tr><th>Tutor activities</th><th>Students activities</th></tr>
<tr><td>Introduction:<br><br>5 minutes</td><td>Work individually, given $h(x)=2x+1$ and $g(x)=2x$ find:<br><br>$f'(x)=....;$<br>$h'(x)=2x+1$<br><br>If<br>$h'(x)=2x+1$<br><br>Find:<br>$h(x)$</td><td>Students work individually.<br><br>Two students, one after another, write the answers on the chalkboard:</td><td>Communication skills developed through the presentation and sharing ideas</td></tr>
<tr><td colspan="4">Development of the lesson</td></tr>
</table>

| | | | |
|---|---|---|---|
| Discovery activity:<br><br>10 minutes | Organizes the students into groups<br><br>Give students activity found in student’s book to discuss in groups and gives instructions related to the task<br><br>Go around to monitor the work of each group and provide assistance where needed | - Studentss form groups.<br><br>-Each group analyses and discusses the proposed activity in student’ s book under the direction of the task manager of the group.<br><br>- Students present to the whole class | **Cooperation** and **communication** skills through discussions<br><br>**Peace and values education**; Cooperation , mutual respect, tolerance through discussions with people with different views and respect one’s views |

| | | | |
|---|---|---|---|
| Presentation of studentss' findings and exploitation:<br><br>15 minutes | Invite the reporter of a sample group to present the findings of the group<br><br>Encourage students to follow attentively others presentations<br><br>Take notes on key points from students' presentation.<br><br>Ask students to amend the presentation and to evaluate their work | -The reporter presents the work on the behalf of the group.<br><br>**Expected answers**<br><br>**(Refer** to solution of activity, in Teacher's guide)<br><br>- Students follow the presentation and ask questions for better understanding.<br><br>- Students evaluate the findings of other students and evaluate their findings. | **Cooperation** and **communication**/ attentive listening during presentations and group discussions<br><br>**Critical thinking** through evaluating other's findings |

| | | | |
|---|---|---|---|
| Conclusion/ Summary:<br><br>5 minutes<br><br>**Assessment**<br><br>**5 minutes** | -Facilitate students to elaborate the summary of the lesson referring to the presentation conducted<br><br>-Request students to write down the main points in their note books and answer their questions.<br><br>- Ask students to individually work out the application activity found in student' s book | - summarize the lesson and come to the main point: Basic integration formulas or immediate integration: they will be put on manila paper and hung on the wall.<br><br>- Students take notes in their books and ask questions for better understanding.<br><br>-Individually, work out the application activity found in student's book and finally make a correction on the chalk board.<br><br>**Expected answers**<br><br>**(**Refer to solution of application activity found in Teacher's Guide). | - **Critical thinking** and problem solving skills are developed through analyzing and solving real life Mathematical problem.<br><br>-**Financial education** is addressed through good management of the school fees brought by the elder brother<br><br>- **Standardization culture**: good habit of paying school fees. |
| **Observation on lesson delivery** | To be completed after receiving the feed-back from the students (what did the students like, what challenged them, and to what extent lesson objective have been achieved) | | |

# PART III: UNIT DEVELOPMENT

# UNIT 1 APPLICATIONS OF MATRICES AND DETERMINANTS

## 1.1 Key unit competence

Apply Matrices and determinants concepts in solving inputs &outputs models and related problems.

## 1.2 Prerequisites

The students will perform well in this unit if they have a good background on

- Matrices and determinants (unit 1 from senior 5)
- Basic concepts of Algebra (Unit1 from senior 4)
- Polynomial functions, equations and inequalities (unit 2 from senior 4)

## 1.3 Cross-cutting issues to be addressed

- Inclusive education (promoting education for all in teaching and learning process)
- Peace and value Education (respect the views and thoughts of others during class discussions)
- Gender: Provide equal opportunity for boys and girls to participate in class

## 1.4 Guidance on introductory activity 1

- Invite students to work in small group, discuss and try out the introductory activity 1 found in student's book;
- Facilitate students' discussions and ask them to avoid noise or other unnecessary conversations;
- During discussions, let students think of different ways to solve the given problem;
- Walk around in all groups to provide assistance where needed;
- Invite group members to present their findings and encourage both boys and girls to actively participate in presentations;

- Let other students share complement on the presentations of their classmates
- Basing on students' experience, prior knowledge and abilities shown in answering the questions for this introductory activity, use different probing questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

## Answer for introductory activity 1

Let the prices of the three commodities $A, B \text{ and } C$ be $x, y \text{ and } z$ per units respectively. By forming system of linear equations we find,

$$\begin{cases} 2x+5z-4y=15000 \\ 3x+y-2z=1000 \\ 3y+z-x=4000 \end{cases}$$

From the system of linear equations let us form matrix $A$ of order 3then find its inverse

$$A=\begin{pmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{pmatrix}$$

$$a_{ij}=\begin{pmatrix} +(1+6) & -(3-2) & +(9+1) \\ -(-4-15) & +(2+5) & -(6-4) \\ +(8-5) & -(-4-15) & +(2+12) \end{pmatrix}$$

$$a_{ij}=\begin{pmatrix} 7 & -1 & 10 \\ 19 & 7 & -2 \\ 3 & 19 & 14 \end{pmatrix}, \quad Adj(A)=\begin{pmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{pmatrix}$$

Let us calculate the determinant of matrix $A$

$$|A|=2(1+6)+4(3-2)+5(9+1)=68$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\frac{1}{68}\begin{pmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{pmatrix}\begin{pmatrix} 15000 \\ 1000 \\ 4000 \end{pmatrix}$$

$$= \frac{1}{68}\begin{pmatrix} 105000+19000+12000 \\ -15000+7000+76000 \\ 150000-2000+56000 \end{pmatrix}$$

$$x = (1/68)(136000) = 2000$$

$$y = (1/68)(68000) = 1000$$

$$z = (1/68)(204000) = 3000$$

## 1.5 List of lessons

| Headings | # | Lesson title/ sub-headings | Learning objectives | Number of periods |
|---|---|---|---|---|
| 1.1 Application of matrices | Introductory activity | | Arouse the curiosity of students on the content of unit 1 | 1 |
| | 1 | Economics applications | Use matrices in solving mathematical problems that involve economical, production and organizational contexts. | 3 |
| | 2 | Financial applications | Use matrices in solving mathematical problems that involve financial and organizational contexts | 2 |

| | | | | |
|---|---|---|---|---|
| 1.2 System of linear equations | 1 | Introduction linear equations | Apply prerequisites on linear equation to form system of linear equation and deduce matrices from system of equations | 3 |
| | 2 | Application of system of linear equations | Apply system of equations and matrices in solving problems related economic and accounting | 3 |
| 1.3. Input-outputs models and Leontief theorem for matrix of order 2. | 1 | Input-outputs models ( $n=2$ ) | Calculate the outputs levels of each of the two given assets to meet a change in final demand | 2 |
| | 2 | Leontief theorem for matrix of order 2. | Apply Leontief theorem to solve finance, economic, and production problems | 3 |
| 1.4 End unit assessment | | | | 1 |

### Lesson 1: Economics applications

#### a) Learning objective:

Use matrices in solving mathematical problems that involve economical, production and organizational contexts.

#### b) Teaching resources:

Student's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as, Microsoft Excel, and internet.

**c) Prerequisites/Revision/Introduction:**

Students will perform well in this lesson if they are skilled enough in the content of Matrices and determinants leant in senior five.

Students will perform well in this lesson if they are skilled enough in Basic concepts of algebra and equations learnt in senior 4

**d) Learning activities**

- Facilitate students to be organized in small group discussions
- Invite Students to work in their respective small group discussions and do the learning activity 1.1.1 from the senior six Mathematics student's book;
- Move around for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class and let other students give their comments on different presentations
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills on the use of matrices to solve problems related to economic
- Ask students to do the application activity 1.1.1 and evaluate whether lesson objectives were achieved to assess their competences.

**Answers for Learning activity 1.1.1**

Let $Q$ be the matrix denoting the quantity of each brand of biscuit bought by $\mathrm{P,Q}$ *and* $R$ and let $C$ be the matrix showing the cost of each brand of biscuit

$$Q=\begin{bmatrix}10 & 7 & 3\\ 4 & 8 & 10\\ 4 & 7 & 8\end{bmatrix},\ C=\begin{bmatrix}4000\\ 5000\\ 6000\end{bmatrix}$$

Since number of rows of first matrix should be equal to the number of columns of the second matrix for multiplication to be possible, the matrices can be multiplied .

$$Q\times C=\begin{bmatrix}10 & 7 & 3\\ 4 & 8 & 10\\ 4 & 7 & 8\end{bmatrix}\begin{bmatrix}4\\ 5\\ 6\end{bmatrix}=\begin{bmatrix}93,000\\ 116,000\\ 99,000\end{bmatrix}$$

Therefore, the amount spent the first customer, second customer and third customer is 93,000FRW; 116,000FRW and 99,000FRW respectively

**e) Answers for the application activity 1.1.1**

$TR-TC=15q_1+18q_2-(2q_1^2+2q_1q_2+3q_2^2)$

$=15q_1+18q_2-2q_1^2-2q_1q_2-3q_2^2$

Condition, $15q_1+18q_2=0$ and $2q_1^2+2q_1q_2+3q_2^2=0$

In matrix form $\begin{bmatrix}-4 & -2\\ -2 & -6\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}=\begin{bmatrix}-15\\ -18\end{bmatrix}$

Solving by Cramer's rule $|A|=\begin{vmatrix}-4 & -2\\ -2 & -6\end{vmatrix}=24-4=20$

$\triangle q_1=|A_1|=\begin{vmatrix}-15 & -2\\ -18 & -6\end{vmatrix}=90-36$, therefore $\triangle q_1=|A_1|=54$

$\triangle q_2=|A_2|=\begin{vmatrix}-4 & -15\\ -2 & -18\end{vmatrix}=72-30$, therefore $\triangle q_2=|A_2|=42$

Thus, $q_1=\dfrac{54}{20}=2.7$ $q_2=\dfrac{42}{20}=2.1$

## Lesson 2: Financial applications

### a) Learning objectives:

Use matrices in solving mathematical problems that involve financial and organizational contexts

### b) Teaching resources:

Student's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers internet connection and mathematical software Microsoft Excel.

### c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they are skilled enough in the content of Matrices and determinants leant in senior five, and the content of previous lesson

Students will perform well in this lesson if they are skilled enough in Basic concepts of algebra and equations learnt in senior 4

### d) Learning activities

- Invite students to work in small groups and do the learning activity 1.1.2 in their Mathematics books;
- Move around in the class for facilitating students where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a teacher, harmonize findings from students' presentation;
- Use different probing questions and guide students to explore the content and examples given in the students' book and lead them to the use of matrices in financial problems
- Invite students to apply these properties in evaluating indefinite integrals
- After this step, guide students to do the application activity 1.1.2 and evaluate whether lesson objectives were achieved.

**Answers for Activity 1.1.2**

Answers will vary according to respondents. As teacher, harmonize the answers from respondents

### e) Answers of application activity 1.1.2

After the visiting the finance office and see how he/she completes the cash books then students create their own matrices of order 3 or less then explain how they formed the matrices.

They create the vector transaction from the created matrices . Teacher harmonizes the answers from students.

## Lesson 3: Introduction to linear systems

### a) Learning objective:

Apply prerequisites on linear equation to form system of linear equation and deduce matrices from system of equations

### b) Teaching resources:

students’s book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

### c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they are skilled enough in the content of Matrices and determinants leant in senior five especially on system of linear equations, and the content of previous lesson

Students will perform well in this lesson if they are skilled enough in Basic concepts of algebra and equations learnt in senior 4

### d) Learning activities

- Ask the students to be organized in small group discussions and do learning activity 1.2.1
- Move around in the class to verify students’ progress over the work.
- Have groups with different activities to present their answers to the whole class.
- As a teacher, harmonize students’ insights by insisting on formulation of system of linear equations and deduce matrices from that system
- Ask students in pairs, to discuss the how to solve the given system of linear equations by using matrices
- Use different probing questions and guide students to explore examples and content from the student book to solve related examples
- After this step, guide the students to complete application activity 1.2.1 conduct, assess students’ competences and whether the objectives have been achieved.

## Answers of learning activity 1.2.1

a) Let consider $x$ as the cost for one cock and $y$ the cost of one Rabbit, the equations that illustrate the activity of Kalisa is $\begin{cases} 5x+4y=35,000 \\ 3x+6y=30,000 \end{cases}$

b) The matrix $A$ indicating the number of cocks and rabbits is $A=\begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix}$

c) In matrix form the activity of Kalisa is written as $\begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 35,000 \\ 30,000 \end{pmatrix}$

d) After discussion students discover that the cost of one cock is 5000FRW and the cost of one Rabbit is 2500FRW

e) Answers to application activity 1.2.1

$$\begin{cases} -x+2y=5 \\ 2x+3y=4 \\ 3x-6y=-15 \end{cases} \Rightarrow [A/B]=\left[\begin{array}{cc|c} -1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & -6 & -15 \end{array}\right] L_2 \sim L_2+2L_1$$

$$\Rightarrow [A/B]=\left[\begin{array}{cc|c} -1 & 2 & 5 \\ 0 & 7 & 14 \\ 3 & -6 & -15 \end{array}\right] L_3 \sim L_3+3L_1$$

$$\Rightarrow [A/B]=\left[\begin{array}{cc|c} -1 & 2 & 5 \\ 0 & 7 & 14 \\ 0 & 0 & 0 \end{array}\right] L_2 \sim \frac{1}{7}L_2 \Rightarrow [A/B]=\left[\begin{array}{cc|c} -1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, $y=2 \quad and \ x=-1$ this gives the set of solution $S=\{(-1,2)\}$

The Gauss helps us to solve the given system

### Lesson 4: Application of system of linear equations

**a) Learning objective:**

Apply system of equations and matrices in solving problems related economic and accounting

**b) Teaching resources:**

**S**tudent's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used

**c) Prerequisites/Revision/Introduction:**

Students will perform well in this lesson if they are skilled enough in the content of Matrices and determinants leant in senior five especially on system of linear equations, and the content of previous lesson.

Students will perform well in this lesson if they are skilled enough in Basic concepts of algebra and equations learnt in senior 4

**d) Learning activities**

- Invite students to work in pairs and do the learning activity 1.2.2 in their Mathematics books;
- Move around in the class for facilitating students in their respective pairs where necessary;
- Verify and identify pairs with different working steps;
- Invite one member from each pairs with different working steps to present their work where they must explain the working steps;
- As a teacher, harmonize their findings from the presentation.;
- Use different probing questions and guide students to explore the content and examples given in the students' book and lead them to discover how to solve economic problems through system of linear equations
- After this step, guide students to do the application activity 1.2.2 and evaluate whether lesson objectives were achieved.

## Answers of learning activity 1.2.2

**Let us summarise the information in the table**

| | Model | | | Time used |
|---|---|---|---|---|
| | X | Y | Z | |
| Juice processing | 30 | 20 | 30 | 2500 |
| Bread processing | 40 | 50 | 30 | 3500 |
| Yoghurt processing | 30 | 30 | 20 | 2400 |

Assign variables to represent the unknowns as follows, $x$ is number of Model $X$ produced, $y$ is number of Model $Y$ produced, and $z$ is number of Model $Z$ produced.

Based on the table, we obtain the following system of equations:

$$\begin{cases} 30x+20y+30z=2500 \\ 40x+50y+30z=3500 \\ 30x+30y+20z=2400 \end{cases} \rightarrow \begin{cases} 3x+2y+3z=250 \\ 4x+5y+3z=350 \\ 3x+3y+2z=240 \end{cases}$$

The augmented matrix of this system given by $\left[\begin{array}{ccc|c} 3 & 2 & 3 & 250 \\ 4 & 5 & 3 & 350 \\ 3 & 3 & 2 & 240 \end{array}\right]$

By row echelon we yield $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 100 \\ 0 & 1 & -1 & -10 \\ 0 & 0 & 1 & 30 \end{array}\right]$

The matrix is now in row echelon form. We find that $z=30$. From row 2, we have $y-z=-10$ so that $y=20$. Finally from row 1, we have $x+3y=100$ so that $x=40$. The solution of the system is $x=40, y=20, z=30$. In one day 40 models X, 20 models Y, and 30 models z processors are produced.

## e) Answers for application activity 1.2.2

Personal savings: $0.05x_1$, Bank loans: $0.06x_2$

Since the couple requires 5,000,000 FRW from these investments, we have the equation

$0.05x_1+0.06x_2=5,000,000$

The total amount available to invest is 60,000,000 which lead to the equation

$$x_1 + x_2 = 60,000,000$$

These two equations form the system

$$\begin{cases} 0.05x_1 + 0.06x_2 = 5,000,000 \\ x_1 + x_2 = 60,000,000 \end{cases}$$

Write the augmented matrix of this system and proceed to row educe and yield

$$\left[\begin{array}{cc|c} 0.05 & 0.06 & 5,000,000 \\ 1 & 1 & 60,000,000 \end{array}\right]$$

by interchanging rows we obtain

$$\left[\begin{array}{cc|c} 1 & 1 & 60,000,000 \\ 0.05 & 0.06 & 5,000,000 \end{array}\right]$$

Using Gauss elimination we get

$$\left[\begin{array}{cc|c} 1 & 1 & 60,000,000 \\ 0 & 0.01 & 20,000,000 \end{array}\right]$$

This means that by multiplying 10 on the second row we get

$$\begin{aligned} x_1 + x_2 &= 60,000,000 \\ 0x_1 + x_2 &= 20,000,000 \end{aligned}$$

by solving the system we get $x_1 = 40,000,000$ and $x_2 = 20,000,000$

Therefore, mount invested Personal savings must be 40,000,000FRW while amount invested from Bank loans must be 20,000,000FRW.

### Lesson 5. Input-outputs models ( $n = 2$ )

#### a) Learning objective:

Calculate the outputs levels of each of the two given assets to meet a change in final demand

#### b) Teaching resources:

Student’s book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with internet connection and mathematical software like Microsoft Excel can be used...

**c) Prerequisites/Revision/Introduction:**

Students will learn better this lesson if they have a good background matrices and determinants learnt in senior five, and enough skills on previous lessons of this unit.

**d) Learning activities**

- Invite students to be organized in small groups and provide clear instructions about the learning activity 1.3.1
- Invite students to use their student's book and discuss on the learning activity 1.3.1
- Switch to each group to ensure all students are actively participating
- Move around in the class for facilitating students where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As teacher, harmonize their answers from presentation;
- Use different probing questions and guide students to explore the content and examples given in the student' s book and lead them to discover how to calculate input-outputs models ( $n = 2$ )
- After this step, guide students to do the application activity 1.3.1 and evaluate whether lesson objectives were achieved.

**Answers of learning activity 1.3.1**

1. $(I - B)X = D$, where $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

By solving we get $X = (I - B)^{-1} D$ where matrix $B$ is known as technology matrix

If $B$ is the technology matrix the following conditions hold

i) The main diagonal elements in $I - B$ must be positive

ii) $|I-B|$ must be positive

2. The technology matrix $B$ **is** $\begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.7 \end{bmatrix}$

$$I-B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.7 \end{bmatrix}=\begin{bmatrix} 0.6 & -0.1 \\ -0.7 & 0.3 \end{bmatrix}$$

$$|I-B|=(0.6)(0.3)-(0.7)(0.1)=0.11$$

Therefore, $(I-B)^{-1}=\dfrac{1}{|I-B|}adj(I-B)=\dfrac{1}{0.11}\begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$

Since the diagonal elements of $(I-B)$ are positive and $|I-B|$ is positive, the system is viable,

Now, $X=\dfrac{1}{0.11}\begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}\begin{bmatrix} 6.8 \\ 10.2 \end{bmatrix}\rightarrow X=\begin{bmatrix} 27.81 \\ 98.90 \end{bmatrix}$

Gross production of commodity A and B are 27.81 and 98.90 tons

### e) Application activity 1.3.1

The technological matrix given by $B=\begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$

$$I-B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}=\begin{bmatrix} 0.6 & -0.1 \\ -0.7 & 0.4 \end{bmatrix}$$

$$|I-B|=(0.6)(0.4)-(-0.7)(-0.1)=0.17$$

$(I-B)^{-1}=\dfrac{1}{0.17}\begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$ where $D=\begin{bmatrix} 50 \\ 100 \end{bmatrix}$

$$(I-B)^{-1}D=\frac{1}{0.17}\begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}\begin{bmatrix} 50 \\ 100 \end{bmatrix}=\begin{bmatrix} 176.5 \\ 558.8 \end{bmatrix}$$

The steel output is 176.5 tons and coal output is 558.8 tons

Total labour days required is 5(steel output) + 2(coal output) =2000 labour days

## Lesson 6. Leontief theorem for matrix of order 2.

### a) Learning objective:

Apply Leontief theorem to solve finance, economic, and production problems

### b) Teaching resources:

Manila paper, Markers, Calculators, student's book and other Reference textbooks to facilitate research, Mathematical models and Internet connection where applicable

### c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good background on matrices and determinants learnt in senior five and perquisites on previous lesson of this unit.

### d) Learning activities

- Organize students into small groups;
- Provide clear instructions and introduce the learning activity by guiding students in the laerning activity.
- Ask students to use the student's book to discuss all questions of the learning activity 1.3.2
- Switch to each group to ensure all students are actively participating
- Call upon groups to present their findings and harmonize their answers;
- Use various probing questions to guide students to explore examples and content related to Leontief theorem for matrix of order two
- Ask students to give examples of where integration by Leontief theorem for matrix of order two can be used
- After this step, guide the students to do the application activity 1.3.2, assessing their competences and evaluating whether the lesson objectives have been met.

**Answers of learning activity 1.3.2**

1. The open Leontief Model is a simplified economic model of a society in which consumption equals production, or input equals output.

Internal Consumption (or internal demand) is defined as the amount of production consumed within industries, whereas External Demand is the amount used outside of industries. In addition, some production consumed internally by industries, rest consumed by external bodies. This model allows us to calculate how much production is required in each industry to meet total demand.

2. $A=\begin{pmatrix}0.4 & 0.02\\ 0.12 & 0.19\end{pmatrix}, \begin{pmatrix}80\\ 200\end{pmatrix}, X=\begin{pmatrix}x\\ y\end{pmatrix}$ we take the matrix equation, and solve the system

$$X = AX + D \to IX - AX = D \to (I-A)X = D \to X = (I-A)^{-1} D$$

$$X=\begin{pmatrix}0.6 & -0.02\\ -0.12 & 0.81\end{pmatrix}^{-1}\begin{pmatrix}80\\ 200\end{pmatrix}=\begin{pmatrix}142.26\\ 267.99\end{pmatrix}$$

Thus, the result is that the agriculture company has to produce 142.26 units and the manufacturing company has produced 267.99 units in order that both can satisfy its own external demand

## e) Answers of application activity 1.3.2

i) The consumption matrix $C=\begin{bmatrix}0.5 & 0.5\\ 0.4 & 0.2\end{bmatrix}$

ii) The Leontief input-output equation relating C to the production vector $\vec{x}$ and the final demand vector $\vec{v}$ given by $\vec{x}=c\vec{x}+\vec{d}$

iii) We need to solve $(I-C)\vec{x}=\vec{d}$

$$I-C=\begin{bmatrix}0.5 & -0.5\\ -0.4 & 0.8\end{bmatrix}\Rightarrow \det(I-C)=0.2$$

$$(I-C)^{-1}=\frac{1}{0.2}\begin{bmatrix}0.8 & 0.5\\ 0.4 & 0.5\end{bmatrix}=5\begin{bmatrix}0.8 & 0.5\\ 0.4 & 0.5\end{bmatrix}=\begin{bmatrix}4 & 2.5\\ 2 & 2.5\end{bmatrix}$$

Therefore, $\vec{x}=(I-C)^{-1}\vec{d}=\begin{bmatrix}4 & 2.5\\ 2 & 2.5\end{bmatrix}\begin{bmatrix}1000\\ 2000\end{bmatrix}=\begin{bmatrix}9000\\ 7000\end{bmatrix}$

## 1.6. Summary of the unit 1

### 1. Use of matrices in economics and finance

To understand money, jobs, prices, monopolies, and how the world works on a daily basis, one must study economics. All across the world, matrices' properties, determinants, and inverse matrices, together with their addition, subtraction, and multiplication operations, are employed to address these real-world issues. In economics, matrices are typically also used in the production process.

### 2. Applications of system of linear equations

When there is more than one unknown and enough information to set up equations in those unknowns, systems of linear equations are used to solve such applications. In general, we need enough information to set up $n$ equations in $n$ unknowns if there are $n$ unknowns. Solving such system means finding values for the unknown variables which satisfy all the equations at the same time . Even though other methods for solving systems of linear equations exist , one method like the use matrices is the more important choice in economics, finance, and accounting for solving systems of linear equations.

Consider any given situation that represents a system of linear equations, consider the following keys points while solving the problem:

i) Identify unknown quantities in a problem represent them with variables

ii) Write system of equations which models the problem's conditions

iii) Deduce matrices from the system

iv) Solve for unknown variables

### 3. Input-Output Models

Let $b_{ij}=\dfrac{a_{ij}}{x_j} \quad i,j=1,2$

$$b_{11}=\frac{a_{11}}{x_1},\ b_{12}=\frac{a_{12}}{x_2},\ b_{21}=\frac{a_{21}}{x_1},\ b_{22}=\frac{a_{22}}{x_2}$$

The equation (1) take the form $b_{11}x_1+b_{12}x_2+d_1=x_1 \quad b_{21}x_1+b_{22}x_2+d_2=x_2$

Rearranged as $\begin{array}{l}(1-b_{11})x_1-b_{12}x_2=d_1\\ -b_{21}x_1+(1-b_{22})x_2=d_2\end{array}$

In matrix form of the above equations is

$$\begin{pmatrix}1-b_{11} & -b_{12}\\ -b_{21} & 1-b_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}d_1\\ d_2\end{pmatrix}$$

$$\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\right\}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}d_1\\ d_2\end{pmatrix}$$

$(I-B)X=D$, where $B=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$ and $I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $X=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$ and $D=\begin{pmatrix}d_1\\ d_2\end{pmatrix}$

By solving we get $X=(I-B)^{-1}D$ where matrix $B$ is known as technology matrix

If $B$ is the technology matrix the following conditions hold

i) The main diagonal elements in $I-B$ must be positive

ii) $|I-B|$ must be positive

**4. Leontief theorem for matrix of order two**

Let $C$ be the consumption matrix, $D$ be the demand vector, and $X$ be the production vector. Then, $CX$ is the internal consumption.

Starting with $\Pr oduction = Total\ demand(\text{int}ernal\ and\ External)$, solve for $X$.

$X=CX+D$, in the open Leontief model $C\ and\ D\neq\begin{pmatrix}0\\ 0\end{pmatrix}$ are given and the problem is to determine $X$ from this matrix equation $X-CX=D$, $(I-C)X=D$

$\mathrm{X}=(I-C)^{-1}D$, if $I-C$ is invertible then, $(I-C)^{-1}$ is then called the Leontief inverse. Therefore, an industry is profitable if the corresponding column in $C$ has sum less than one.

## 1.7 Additional Information for Teacher

For the teacher to be effective (in order to respond to all aspects of the students' needs), it is worth mentioning that the teacher needs a wide range of skills, attitudes and values, a rich and deep understanding of the mathematics subject matter and the pedagogical processes to develop the understanding that is required from the student. It is therefore, imperative for the teacher to not limit himself/herself to the only to the present book, but also to consider getting information from other relevant books, such as those mentioned in the reference. Here the teacher has to emphasize the application of matrices and determinants in solving problems related to economics, accounting, production and in real life situations. In addition, teachers should be aware of economics terms for better found more applications of matrices and determinants.

## 1.8 End unit assessment

1. Given $\begin{cases} x+2y-3z=0 \\ 3x+3y-z=5 \\ x-2y+2z=1 \end{cases}$ , $AX=B \Rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 3 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}$

**a.** $\begin{bmatrix} 1 & 2 & -3 \\ 3 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$

**b.** $\Delta=\begin{vmatrix} 1 & 2 & -3 \\ 3 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix}=[6-2+18]-[-9+2+12]=17$

$$\Delta_x=\begin{vmatrix} 0 & 2 & -3 \\ 5 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix}=[0-2+30]-[-9+0+20]=17$$

$$\Delta_y=\begin{vmatrix} 1 & 0 & -3 \\ 3 & 5 & -1 \\ 1 & 1 & 2 \end{vmatrix}=[10+0-9]-[-15-1+0]=17$$

,

$$\Delta_z=\begin{vmatrix} 1 & 2 & 0 \\ 3 & 3 & 5 \\ 1 & -2 & 1 \end{vmatrix}=3+10+0-(0-10+6)=17$$

$$x = \frac{\Delta_x}{\Delta} = \frac{17}{17} = 1,\ y = \frac{\Delta_y}{\Delta} = \frac{17}{17} = 1,\ z = \frac{\Delta_z}{\Delta} = \frac{17}{17} = 1$$

$$S = \{(1,1,1)\}$$

**c.** By using other methods obtained we get same values in (b) to the solutions

2. Let $x, y$ *and* $z$ denote the number of cars that can be produced of each type. Then we have,

$$\begin{cases} 2x + 3y + 4z = 29 \\ x + y + 2z = 13 \\ 3x + 2y + z = 16 \end{cases}$$

The information can be represented in matrix form as

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 29 \\ 13 \\ 16 \end{pmatrix}$$

This matrix can be solved by using inverse matrix, Cramer's method or Gauss elimination. By Gauss the final matrix will be

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \\ -20 \end{pmatrix} \rightarrow \begin{cases} x + y + 2z = 13 \\ y = 3 \\ -5z = -20 \end{cases}$$

Hence $z = 4,\ y = 3,\ x = 2$

Therefore, solution is $S = \{(2,3,4)\}$

The numbers of cars of each type which can be produced are 2, 3, and 4 respectively.

3. Let the production level of the three products be x, y and z respectively. Therefore,

$$\begin{cases} x + y + z = 45 \\ -x + 0y + z = 8 \\ x - 2y + z = 0 \end{cases} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 45 \\ 8 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 6$$
,

By calculating $\triangle_x, \triangle_y, \triangle_z$

We get $\triangle_x = 66, \triangle_y = 90, \triangle_z = 114$

$$x = \frac{66}{6} = 11,\ y = \frac{90}{6} = 15,\ z = \frac{114}{6} = 19$$

Hence, the production levels of the products are as follows:

First product is 11 tons, the second product is 15 tons and the third product is 19 tons.

4. a. the consumption matrix C for this economy is given by

$$C = \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.6 \end{bmatrix}$$

b. $I - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 6/10 & -2/10 \\ -2/10 & 4/10 \end{bmatrix}$

$$(I - C)^{-1} = \frac{1}{\left(\frac{6}{10}\right)\left(\frac{4}{10}\right) - \left(\frac{-2}{10}\right)\left(\frac{-2}{10}\right)} \begin{bmatrix} 6/10 & -2/10 \\ -2/10 & 4/10 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

**d.** let $\vec{d} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$,

The Leontief model states that the product vector $\vec{x}$ must satisfy $c\vec{x} + \vec{d}$ or equivalent to $\vec{x} = (I - C)^{-1}\vec{d} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$

Therefore, the production level need to meet the demand is 50 units from mining and 50 units from electricity.

## 1.9 Additional activities

### 1.9.1 Remedial activities

Solve the following system of linear equations

$$\begin{cases} 3x+2y+4z=-1 \\ 2x-y+2z=-2 \\ -x+y+2z=2 \end{cases}$$

**Solution**

1) $\left(\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 2 & -1 & 2 & -1 \\ -1 & 1 & 1 & 2 \end{array}\right) L_2 \sim 3L_2 - 2L_1$

$$\Rightarrow \left(\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 0 & -7 & -2 & -2 \\ -1 & 1 & 1 & 2 \end{array}\right) L_3 \sim 3L_3 - L_1$$

$$\Rightarrow \left(\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 0 & -7 & -2 & -4 \\ 0 & 5 & 10 & 5 \end{array}\right) L_3 \sim 5L_2 + 7L_3$$

$$\Rightarrow \begin{pmatrix} 3 & 2 & 4 & -1 \\ 0 & -7 & -2 & -4 \\ 0 & 0 & 60 & 15 \end{pmatrix}$$

$$\Rightarrow 60z = 15 \Rightarrow \boldsymbol{z} = \frac{1}{4}$$

$$3x+2y+4z=-1 \Rightarrow 3x+1+1=-1 \Rightarrow 3x=-3 \Rightarrow \boldsymbol{x}=-1$$

$$\Rightarrow -7y-2z=-4 \Rightarrow -7y-\frac{2}{4}=-4 \Rightarrow \boldsymbol{y}=\frac{1}{2}$$

$$S=\left\{-1; \frac{1}{2}; \frac{1}{4}\right\}$$

### 1.9.2 Consolidation activities

A firm produces three products A,B and C requiring the mix of three materials P, Q, and R. The requirement (per unit) of each product for each material is as follows

$$M=\begin{pmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 4 & 2 \end{pmatrix}$$

Using matrices notations, find

a) The total requirement of each material if the firm produces 100 units of each product

b) The per unit cost of production of each product if the per unit cost of materials P, Q and R is 5 dollars, 10 dollars, 5 dollars respectively

c) The total cost of production if the firm produces 200 units of each product

**Solution**

a) Total requirement of each material if the firm produces 100 units of each product given by

$$\begin{bmatrix} 100 & 100 & 100 \end{bmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 4 & 2 \end{pmatrix}=\begin{bmatrix} 800 & 900 & 800 \end{bmatrix}$$

b) Let the per unit cost of material be $C=\begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}$

The per unit cost of production of each product would be calculated as

$$AC=\begin{pmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 4 & 2 \end{pmatrix}\begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}=\begin{bmatrix} 45 \\ 65 \\ 60 \end{bmatrix}$$

c) The total cost of production if its firm produces 200 units of each product would be given

$$\begin{bmatrix}200 & 200 & 200\end{bmatrix}\begin{bmatrix}45\\65\\60\end{bmatrix}=34,000$$

Therefore, the total cost of production will be 34,000 dollars

### 1.9.3 Extended activities

An economy has two sectors, electricity and services. For each unit of output, electricity requires 0.5 units from its own sector and 0.4 units from services. Meanwhile, services require 0.5 units from electricity and 0.2 from its own sector to produce one unit of services.

a) Determine the consumption matrix C

b) State the Leontief input-output equation relating C to the production vector and final demand vector

c) Use inverse matrix to determine the production vector necessary to satisfy demand of 1000 units of electricity and 2000 units of services

**Solution**

a. $$C=\begin{bmatrix}0.5 & 0.5\\0.4 & 0.2\end{bmatrix}$$

b. $$\vec{x}=C\vec{x}+\vec{d}$$

c. $$I-C=\begin{bmatrix}0.5 & -0.5\\-0.4 & 0.8\end{bmatrix}\rightarrow \det(I-C)=0.2$$

$$(I-C)^{-1}=\frac{1}{0.2}\begin{bmatrix}0.8 & 0.5\\0.4 & 0.5\end{bmatrix}=\begin{bmatrix}4 & 2.5\\2 & 2.5\end{bmatrix}$$

$$\vec{x}=(I-C)^{-1}\vec{d}=\begin{bmatrix}4 & 2.5\\2 & 2.5\end{bmatrix}\begin{bmatrix}1000\\2000\end{bmatrix}=\begin{bmatrix}9000\\7000\end{bmatrix}$$

# UNIT 2 LINEAR INEQUALITIES AND THEIR APPLICATION IN LINEAR PROGRAMMING PROBLEMS

## 2.1. Key unit competence: Solve linear programming problems

## 2.2. Prerequisites

In this unit, Student must be skilled in previous Units for S4, and 1st Unit of S5,

**i.e.,**

- Solve problems related to linear inequalities and represent the solutions on graph.
- Solve problems involving linear inequality functions and interpret the graphs of those functions.

## 2.3. Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)
- Standardization culture (regulation of the current intensity)

## 2.4. Guidance on introductory activity

- In groups, facilitate students read and do the introductory activity from Students' book
- Facilitate discussions to avoid noise or other unnecessary conversation,
- Be aware of straggling Students.
- Call them to present their findings and promote gender into presentation.
- Through question-answer, facilitate Students to realize that introductory activity stimulates them to get idea on Unit 2.
- Through class discussions, let Students think on different ways of getting solutions.

## 2.5. List of lessons

| Headings | # | Lesson title/ sub-headings | Learning objectives | Number of periods |
|---|---|---|---|---|
| 2.1. Recall linear inequalities, | Introductory activity 2 | | Arouse the curiosity of students on the content of unit 2. | 1 |
| | **1** | Problems involves linear inequalities | Apply inequalities to solve problems in finance, economics and production | 2 |
| **2.2** Basic concepts of linear programming problem (LPP) | **1** | **Definition and keys concepts:** objective function, decisions variables, constraints. | Solving simultaneous equations using graphical, elimination, comparison, and substitution methods | 2 |
| | **2** | Mathematical models formulation and optimal solution. | Decide the appropriate optimal solutions and draw conclusions | 3 |
| **2.3** Method of solving LPP | 1 | Graphical method | Apply linear inequalities to solve linear programming problem | 5 |
| **2.4 End unit assessment** | | | | **2** |
| | | | | 15 |

### Answer of introductory activity 2:

1. (a) Answer on this question will vary from a Student to another. Teacher shall see the Cartesian plane drawn by each student and assess the level of understanding to formulate linear inequalities, system of linear inequalities, and how they represent them on Cartesian plane. As a teacher, verifies their truthfulness and guides them accordingly.

(b) Teacher will see how students use constraints on the Cartesian plane. The linkage is about to draw the constraints on the Cartesian plane, might be inequalities or equations.

2. Reference is made to the provided data as shown in the following table:

| TYPES OF COMPANY | High-quality | Medium-quality | Low-quality | Cost ("000") |
|---|---|---|---|---|
| A | 600 | 300 | 400 | 180 |
| B | 100 | 100 | 600 | 160 |
| Availability capacity (Dozens) | 1200 | 800 | 2400 | |

Let $x$ be the quantity produced by company A, and $y$ be the quantity produced by company B.

Objective function is to üüüüüüüüüüüü$(\ ) = \ \ , \ \ x + \ \ , \ \ y$

And the Constraints are:

$$\begin{cases} 600x + 100y \leq 1200 \\ 300x + 100y \leq 800 \\ 400x + 600y \leq 2400 \end{cases}$$

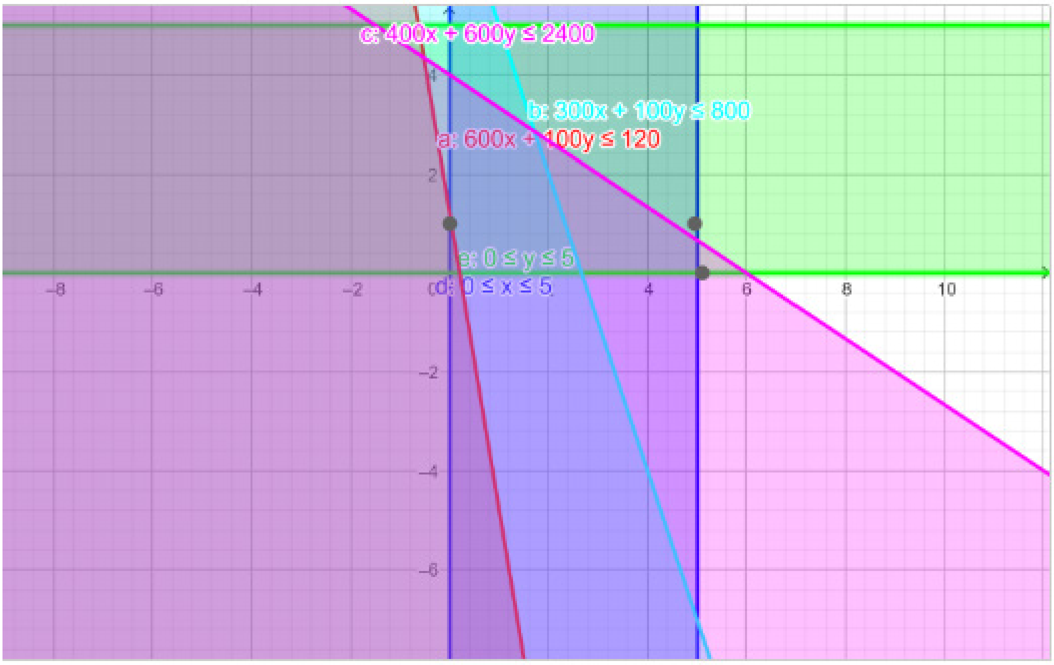
$$0 \leq x \leq 5, \text{ and } 0 \leq y \leq 5$$

Compute the full results accordingly, but hereafter are the guidance you may follow in class.

The feasible region is above the following two equations: $\begin{cases} 400x + 600y = 2400 \\ 300x + 100y = 800 \end{cases}$,

And the solution of this system is one of the corner points which is the optimal, at point (.67, 5) the Cost =920,000FRW; at (1.7, 2.9) the cost =772,571.4FRW; at (1, 5), the cost = 980,000FRW.

Then, the optimal point is (1.7, 2.9). Reference is made to the graph shown hereafter, details all steps while explaining to the students, and show all those feasible region and feasible points on the board.




### Lesson 1: Recall of linear inequalities

**a) Learning objective: Solve linear inequalities**

**b) Teaching resources:**

- Student's book,
- Reference books.
- Ruler, T-square, Manila paper
- Scientific calculators.
- Internet connection

**Note:** If possible computers with potential mathematical tools and software, these include Geogebra, Microsoft Excel, Math-lab or Graph-Calc and internet can be used.

**c) Prerequisites / Revision / Introduction:**

Students will learn better this lesson if they have a good understanding on taught matter in Senior 4. It is noted that students should to be skilled on:

- Representation and interpretation of graphs of linear functions
- Representing linear equations on Cartesian plane
- Performing operations, factorizing polynomials, etc.

**d) Learning activities**

- Organize the students into small groups;
- Provide clear instructions and introduce the learning activity by guiding students.
- Teacher asks learners to use the student book to discuss on learning **activity 2.1.**
- Move around to ensure all students in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings and facilitate them to summarize the concepts related to this lesson.
- Guide students to perform individually **application activity 2.1** to assess their knowledge and skills.

**Answers of learning activity 2.1**

1. a) $x<5$ represents all numbers less than 5, for example 4, 3, 2, 1, 0, -1, -2, etc.

   b) $x>0$ Represents all numbers greater than 0, for example 1, 2, 3, 4, 5, 6 etc

   c) $-4<x<12$ includes for example -3,-2,-1,0,1,2,3,...,11**.**
2. Response on this question will vary from a group to another, as a teacher, try to verify their truthfulness and correct them accordingly.
3. Let A stands for Alex, and S stands for Sam

$A=S+3;\Rightarrow S+A<9;\Rightarrow S+S+3<9;\Rightarrow 2S<9-3;\Rightarrow 2S<6;\Rightarrow S<3;$ Sam could score goals which are less than 3. Therefore could score 0, 1 or 2 goals. If Sam score 0, Alex score 0+3=3, if Sam score 1 goal, Alex score 1+3=4 and if Sam score 2 goals Alex score 2+3=5 goals. Then, Alex could score 3, 4 or 5 goals.

**e) Answer of application activity 2.1.**

1. Let $S$ be the average running speed; and the average cycling speed is $2S$

$Speed=\frac{Dis\tan ce}{Time}$; $The\ time\ is<2\frac{1}{2}$;Therefore,

$$\frac{25}{2S}+\frac{20}{S}<2\frac{1}{2}\Rightarrow 25+24<2S\times 2\frac{1}{2}\ 25+40<2S\times 2\frac{1}{2}$$

$$\Leftrightarrow 25+40 < 2S \times \frac{5}{2} \Rightarrow 65 < 5S\,; S > \frac{65}{5} \Rightarrow S > 13$$

So, her average running is greater than $13km/h$, and her average speed cycling is greater than $26km/h$

2. The answers vary according to the answers given by students, teacher shall harmonize the answers.

## Lesson 2: Definition and keys concepts of linear programming problem (LPP)

### a) Learning objective:

Solve linear inequalities

### b) Teaching resources:

- Student's book,
- Reference books.
- Ruler, T-square, Manila paper
- Scientific calculators.
- Internet connection

**Note:** If possible computers with potential mathematical tools and software, these include Geogebra, Microsoft Excel, Math-lab or Graph-Calculator and internet can be used.

### c) Prerequisites / Revision / Introduction:

Students will learn better this lesson if they have a good understanding on taught matter in Senior 4, and lesson 1 for this unit. It is noted that students should to be skilled on:

- Representation and interpretation of graphs of linear functions
- Representing linear equations on Cartesian plane
- Performing operations, factorizing polynomials, etc.

### d) Learning activities

- Organize the students into small groups;
- Provide clear instructions and introduce the activity by guiding students.
- Teacher asks learners to use the student book to discuss on learning **activity 2.2.1**

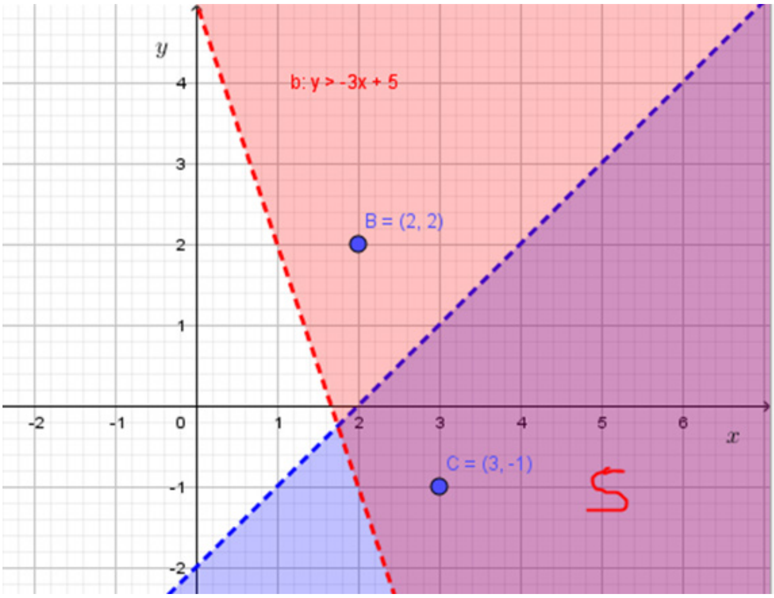
- Move around to ensure all students in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings and facilitate them to summarize the concepts related to this lesson.
- Guide students to perform individually **application activity 2.2.1** to assess their knowledge and skills.

### Answers for learning activity 2.2.1

1. The points which verify the system of the system of inequalities shown below

$\begin{cases} y < x - 2 \\ y > -3x + 5 \end{cases}$ Are $B(2,2)$, and $C(3,-1)$.

The best solution is the intersection of the shaded regions as shown on the following graph.



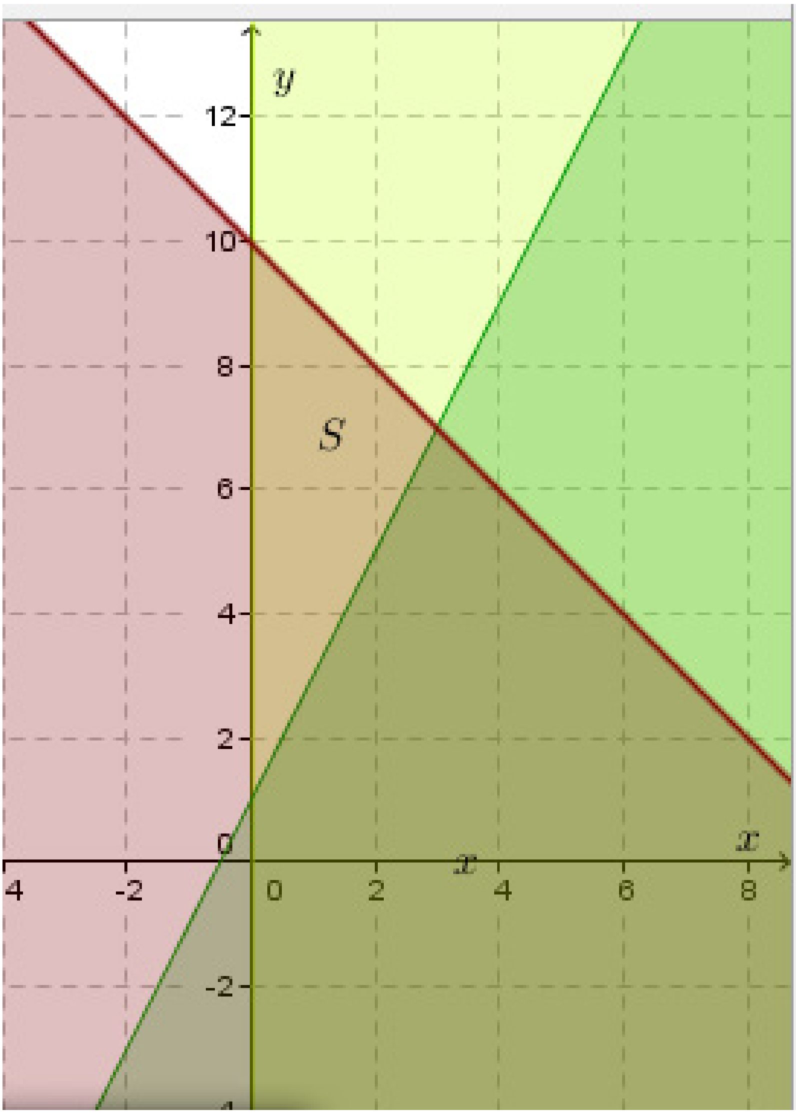

2. The blue is $y < x - 2$. It is clear that the point B (2,2) is in the region solution of the first inequality but it is not in the region solution of the second inequality.
3. The intersection of the shaded regions is the solution region S. This is the reason why the point C (3,-1) verifies all inequalities.
4. The role of linear programming model in investments and businesses are interested on maximizing profits, or minimizing the total costs. This is an open questions teacher will read the idea of students and awarded each accordingly.

5. Yes, there is a linkage between inequalities and linear programming. Can be combined under mathematical model from the abstract model. But consider idea of student too.

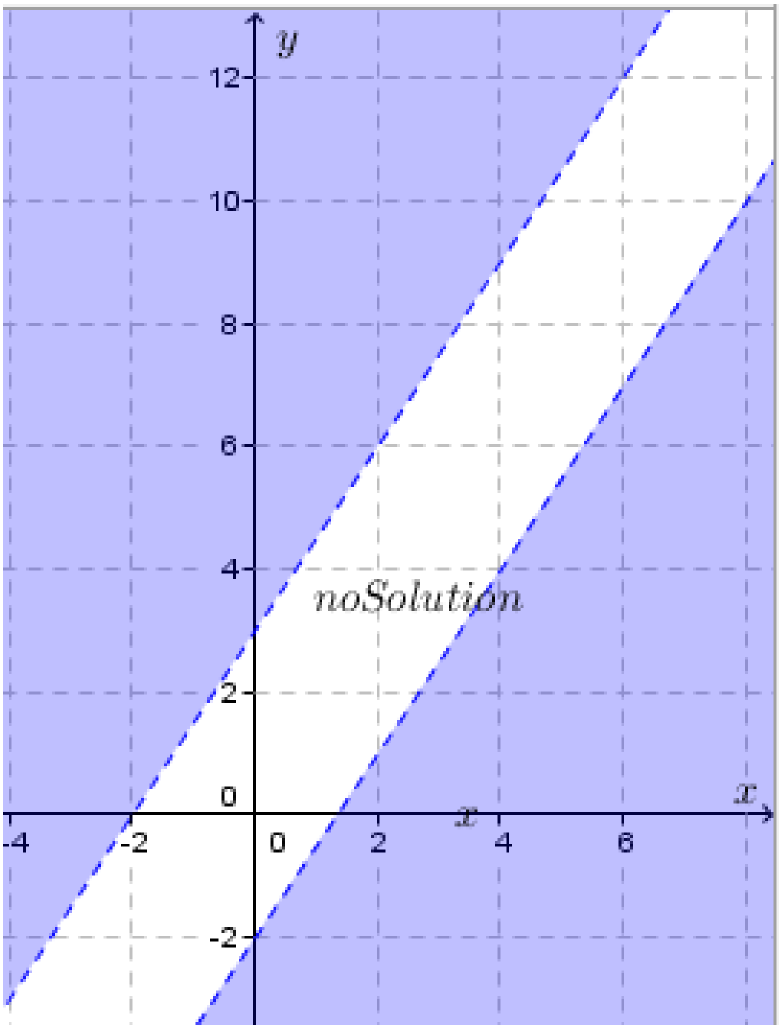
**e) Answers for application activity 2.2.1**

1. 1)

a)




Then, S is in the part of which represents the solution




2.

i) Create any linear inequalities related to this problem;

- $50A + 70B \geq 970$ Passenger constraint
- $40A + 25B \geq 370$ Baggage constraint
- $A \geq 0, B \geq 0$ Non-negative constraint

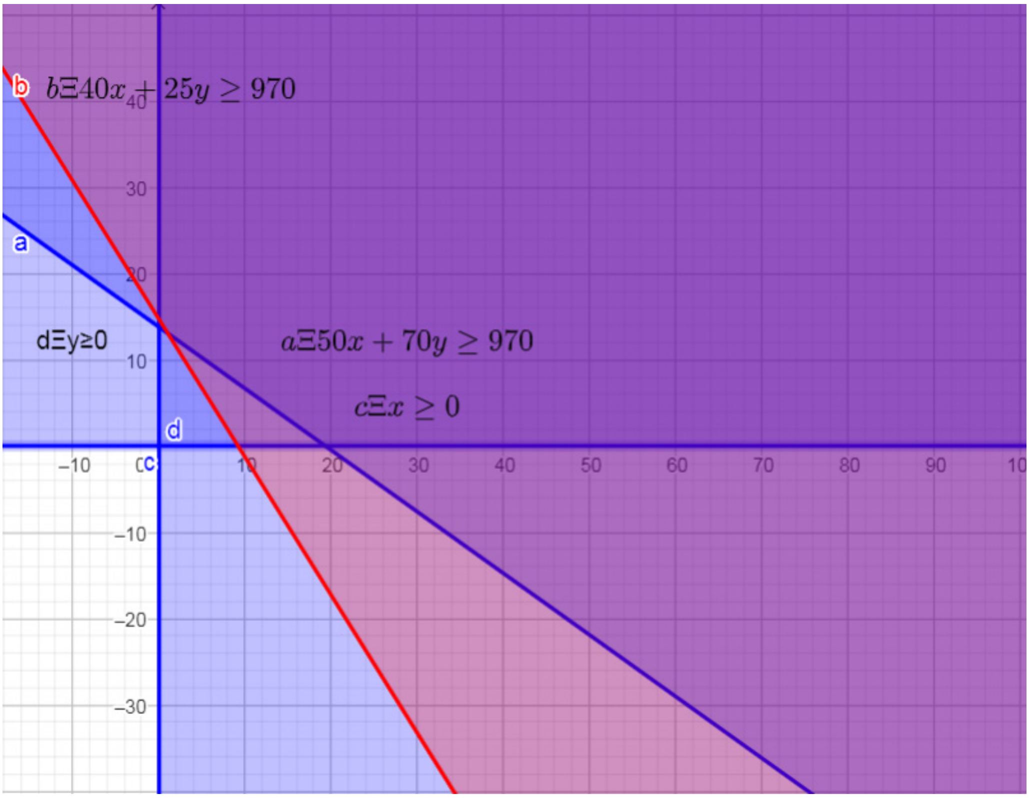
ii) What are your observations in terms of graphical method?

- The above mentioned linear inequalities can be presented on the graph in order to find the feasible region which might help us to find the optimal solution.

iii) Is there any purpose? If so, which one?

- Yes, we have to Minimize $Z = 1{,}100\ A + 1{,}300B$; where Z = total cost (RWF)

iv) Establish the variables under consideration.




x-axis represent bus B, and y-axis represent bus A. Students might use other notation; you are required to re-consider accordingly.

### Lesson 3: Mathematical models formulation and optimal solution in LPP

#### a) Learning objective:

Formulate a mathematical model from abstract model

#### b) Teaching resources:

- Student's book,
- Reference books.
- Ruler, T-square, Manila paper
- Scientific calculators.
- Internet connection

**Note:** If possible computers with potential mathematical tools and software, these include Geogebra, Microsoft Excel, Math-lab or Graph-Calc and internet can be used.

**c) Prerequisites / Revision / Introduction:**

Students will learn better this lesson if they have a good understanding on taught matter in Senior 4. And lesson 1and 2 for this unit. It is noted that students should be skilled on:

- Representation and interpretation of graphs of linear functions
- Representing data in Cartesian plane.
- Performing operations, factorizing polynomials, etc.

**d) Learning activities**

- Organize the students into small groups;
- Provide clear instructions and introduce the activity by guiding students.
- Teacher asks students to use the student book to discuss on **activity 2.2.2**
- Move around to ensure all students in groups participate actively.
- Call upon groups to present their findings.
- Harmonize their findings and facilitate them to summarize the concepts related to this lesson.
- Guide students to perform individually **application activity 2.2.2** to assess their knowledge and skills.

**Answer of learning activity 2.2.2.**

I think this can be computed instead, without computation might cause a highest level of erroneous. That is why computation are recommended. Guide students accordingly in this logic framework.

i) How can you reduce expenses?

**We** can set out the problem as a linear programme, Minimize Z = 1,100A + 1,300B; Where Z = total cost (“000”FRW)Subject to:

$50A + 70B \geq 970$ Passenger constraint

$40A + 25B \geq 370$ Baggage constraint

$A \geq 0, B \geq 0$ Non-negative constraint

The following figure graphs these passenger and baggage constraints as equalities.

**Graphing constraints as equalities:**

Passenger constraint: 50A + 70B = 970; and Baggage constraint: 40A + 25B = 370

Can be presented on the following graph:


Reference is made to the above graph; the value of the objective Z will be at a minimum at one or more of the corner points as shown below.

- Value of Z at (A = 19.4, B = 0)

$$Z = 1,100,000(19.4) + 1,300,000(0) = \mathbf{21,340,000FRW}$$

- Value of Z at (A = 0, B = 14.8)

$$Z = 1,100,000(0) + 1,300,000(14.8) = \mathbf{19,240,000\ FRW}$$

- **Value of Z at intersection of the equations:** $\begin{cases} 50x + 70y = 970 \\ 40x + 25y = 370 \end{cases}$ give us the coordinate of the intersection. Here the co-ordinates are, unknown unless our graph is drawn precisely so that we can read off the co-ordinates directly from the graph. Even then, we can use simultaneous equations to solve the co-ordinate is (1.1, 13.1) therefore, we have two lines which intersect at this point, i.e. which have the same (simultaneous) values of $A = x$ and $B = y$ at this point.

1. 50A + 70B = 970 Passenger constraint
2. 40A + 25B = 370 Baggage constraint

Multiplying equation (2) by 2.8 gives equation (3)

3. 112A + 70B = 1,036

**(1) – (3)** - 62A = - 66

A = -66/-62 = 1.1

Substitution A = 1.1 in equation (1)

50(1.06) + 70B = 970

70B = 970 – 53 = 917

B = 13.1

So the co-ordinates of corner point is the intersection which is, A = 1.1, B = 13.1

◊ **The value of Z at the intersection point is**

Z at intersection = 1,100(1.1) + 1,300(13.1) = **18,240,000 FRW**

Clearly, the volcanoes Express Company can minimize the total cost by providing 1.06 vehicles of type A and 13.1 vehicles of type B per journey. This will cost volcanoes Express Company Ltd 18,240,000 FRW which is the lowest feasible total cost given that 970 passengers and 370 tons of baggage must be carried per journey.

### e) Answer of application activity 2.2.2

1. Let the two kinds of instruments be such that there are $x$ number of the first instrument and $y$ number of the second instrument. Given that 9 hour of fabrication time and 1 hour of finishing time is needed for each of the $x$ number of the first instrument. Also 12 hours of fabricating time and 2 hours of finishing time is required for $y$ number of the second instrument. Further, there are a total of 180 hours for fabricating and 30 hours for finishing. These can be defined as the two constraints.

   **Constraint - 1(For Finishing):** $9x + 12y < 180 \ or 3x + 4y < 60$

   **Constraint - II(For Fabricating):** $x + 2y < 30$.

   The company makes a profit of 800,000Frw on each of the x numbers of the first instrument, and a profit of 1200,000Frw on each of the y numbers of the second instrument. The aim is to maximize the profits and this can be represented as an objective function.

**Objective Function:** $Z = 800,000x + 1200,000y$

Therefore the two constraints are $3x + 4y<60,\ x + 3y<30$, and the objective function is $Z = 800,000x + 1200,000y$.

## Lesson 4: Solving LPP using graphical methods

### a) Learning objective:

Solve LPP using graphical method

### b) Teaching resources:

- Student's book,
- Reference books.
- Ruler, T-square, Manila paper
- Scientific calculators.
- Internet connection

**Note:** If possible computers with potential mathematical tools and software, these include Geogebra, Microsoft Excel, Math-lab or Graph-Calc and internet can be used.

### c) Prerequisites / Revision / Introduction:

Students will learn better this lesson if they have a good understanding on taught matter in Senior 4. And lessons 1,2,3 for this unit. It is noted that students should to be skilled on:

- Representation and interpretation of graphs of linear functions
- Representing data in Cartesian plane.
- Performing operations, factorizing polynomials, etc.

### d) Learning activities

- Organize the students into small groups;
- Provide clear instructions and introduce the activity by guiding students.
- Teacher asks learners to use the student book to discuss on learning **activity 2.3.1**
- Move around to ensure all students in groups participate actively.

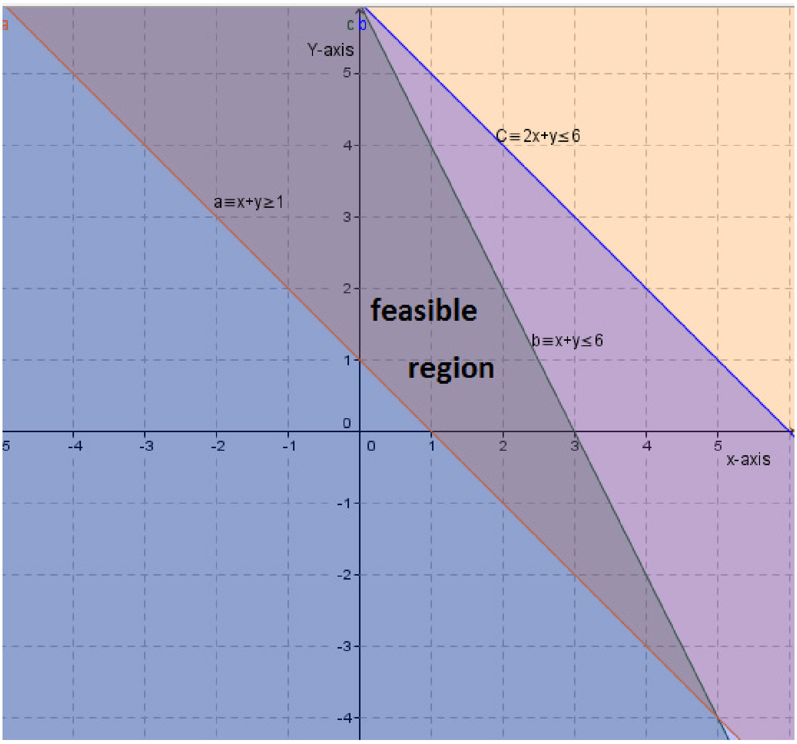
- Call upon groups to present their findings.
- Harmonize their findings and facilitate them to summarize the concepts related to this lesson.
- Guide students to perform individually **application activity 2.3.1** to assess their knowledge and skills.

**Answer of learning activity 2.3.1**

a. the steps can be followed to find solution are: (let us consider steps from student book)

b. The graph of the following system of inequality:

$$\begin{cases} x+y \geq 1 \\ x+2y \leq 6 \\ 2x+y \leq 6 \\ x \geq 0, y \geq 0 \end{cases}$$




The reader should take note of the first constraint $x+y \geq 1$, the feasible region must be bounded below by the line $x+y=1$; the test point (0,0) does not satisfy $x+y \geq 1$; so we shade the region on the opposite side of the line from test point (0,0).

c. Yes, profit can be shown from the graph. check critical points and

income below.

| Critical points | Income |
|---|---|
| (1,0) | 10(1)+15(0)=\$10 |
| (3,0) | 10(3)+15(0)=\$30 |
| (2,2) | 10(2)+15(2)=\$50 |
| (0,3) | 10(0)+15(3)=\$45 |
| (0,1) | 10(0)+15(1)=\$15 |

The corner points (the critical points) listed on the figure are (0,3),(0,1),(1,0),(3,0), and (2,2) and the optimal point is the point (2, 2) which maximizes the objective function to a maximum value of 50USD. It is important to note that if the point (0,0) is on the line for a constraint, it cannot be used as a test point. In that case, we would need to choose any other point that is not on the line to use as a test point.

### e) Answer of application activity 2.3.1

Let $x$ is the number of hours per week Aline will work at storekeeper; and $y$ is the number of hours per week Aline will work at accountant occupations respectively. First of all, we have to write the objective function, and it is given that Aline gets paid \$40 an hour at storekeeper, and \$30 an hour at accountant, her total income is denoted as $I$ is given by the following equation. $I = 40x + 30y$

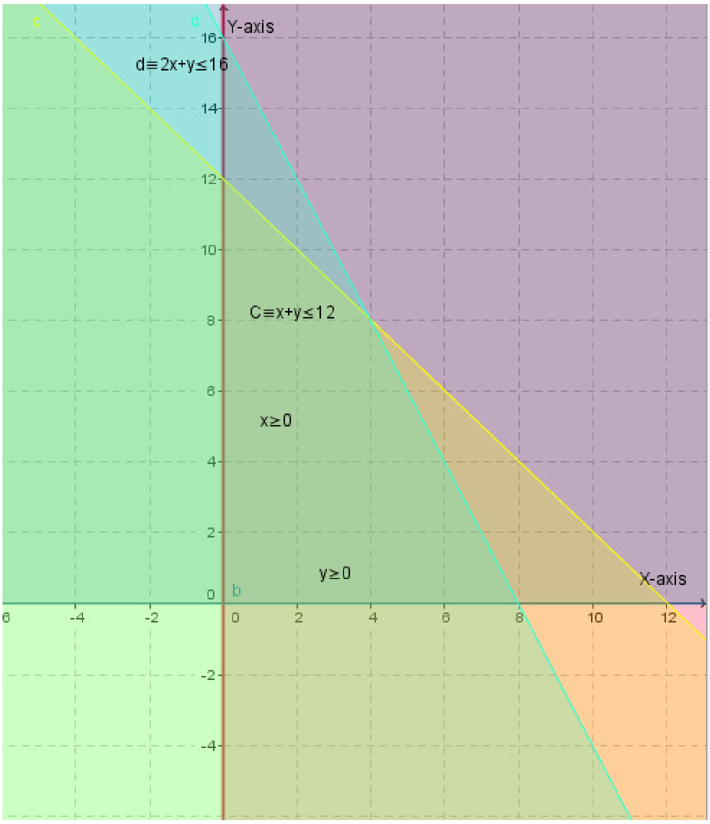
Then, next step, we find the constraints/conditions; basing on the sentence states in the question, "she never wants to work more than a total of 12 hours a week" can be translated to mathematical constraint: $x + y \leq 12$

Also, another sentence states, "for every hour she works at Storekeeper, she needs 2 hours of preparation time, and for every hour she works at Accountant, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation", can be written as: $2x + y \leq 16$

These two inequalities show the technological constraints while the fact that $x$ and $y$ can never be negative is represented by the following two constraints: $x \geq 0$, and $y \geq 0$. We have to formulate the mathematical model for this problem as follows:

$$
\begin{aligned}
&Maximize I = 40x + 30y \\
&subject \quad to \\
&\begin{cases} x + y \le 12 \\ 2x + y \le 16 \\ x, y \ge 0 \end{cases}
\end{aligned}
$$

To solve this problem, we plot all the constraints and shade the region that meets all of the inequality constraints. The lines for the constraints can be graphed using any appropriate method. The simplest method, however, is to graph the line by plotting the x-intercept and y-intercept. A constraint line will divide the plane into two regions, one of which will satisfy the inequality part of the constraint. To satisfy the inequality, a test point is used to determine which portion of the plane to shade. A test point can be any point on the plane that is not on the line. If the test point satisfies the inequality, then the region of the plane containing the test point satisfies the inequality. If the test point does not satisfy the inequality, the region that does is on the other side of the line from the test point. After the lines representing the constraints were graphed using an appropriate method in the graph below, the point (0,0) was used as a test point to determine that (0,0) satisfies the constraints under consideration because $0+0<12$ , and $2(0)+0<16$ , therefore, we have to shade the region that is below and to the left of both constraint lines, but also above the x axis and to the right of the y axis, in order to further satisfy the constraints $x \ge 0$, and $y \ge 0$.




The intersection of these two line is (4,8),is obtained by solving the system of two equations, therefore, the feasible region is lying between these four corners points (0,12), (4,8), (8,0), and (0,0). All conditions are met in the feasibility region or polygon. According to the Fundamental Theorem of Linear Programming, the maximum (or minimum) value of the objective function always occurs at the vertices of the feasibility region. To maximize Aline's income, we'll use the objective function to see which point gives us the highest weekly income on these critical points must be (0, 0), (0, 12), (4, 8), (8, 0). The outcomes are listed below:

| **Critical Points** | Income |
|---|---|
| (0, 0) | 40(0) + 30(0) = \$0 |
| (0, 12) | 40(0) + 30(12) = \$360 |
| **(4, 8)** | **40(4) + 30(8) = \$400** |
| (8, 0) | 40(8) + 30(0) = \$320 |

Basing on these results, the point (4, 8) gives the most profit: \$400. Therefore, Aline should work 4 hours at Storekeeper, and 8 hours at Accountant.

## 2.6. End Unit Assessment

### Answers of End Unit Assessment

**Answer 1:**

i) Let $x$ be the number of business class tickets sold, and let $y$ be the number of economy class tickets sold. It is given that the airline earns a total profit of 320 for each business class ticket and a profit of 400 for each economy class ticket. This information can be written in the form of following equation: $P = 320x + 400y$. Basing on the given scenario, the feasible region is near (25,155), (90,90), and (25,90) because all these are corners points, that is why the optimal can be considered as: $x+y \le 180 \quad \Rightarrow x \ge 25, y \ge 90$ *therefore, the value for* $x = 25, y = 155$, using the following mathematical model, with the objective function, and constraints:

$$Max\ P = 320x + 400y$$

$$st.\begin{cases} x+y \le 180 \\ x \ge 25 \\ y \ge 90 \end{cases}$$

The airline must sell at least 25 business class and 90 economy class tickets to be profitable, to improve the explanations, teacher shall clearly help students to handle this in class.

ii) $P = 320x + 400y$, to get the maximum profit that can be earned by the airline: $P = 320x + 400y = 320(25) + 400(155) = 8,000 + 62,000 = 70,000\,U$

Other points might provide smaller profit compared to this, which are 44,000U and 64,600U respectively, that is why the optimal point is $x = 25,\ y = 155, \quad P = 70,000U(Money)$

**Answer 2:**

i) Follow the following steps to solve the given problem:

**Step 1** - Identify the decision variables, let us suppose that $x$ be the number of caramel cookies sold daily. And let $y$ be the number of chocolate chip cookies sold daily.

**Step 2** - Write the Objective Function, since each chocolate chip cookie yields the profit of $0.75 and each caramel cookie produces a profit of $0.88, therefore we will write the objective function as: $P = 0.88x + 0.75y$, where x and y are non negative values.

**Step 3** - Identify set of constraints, it is mentioned in the problem that the demand forecast of caramel cookies is at least 80 and the bakery cannot produce more than 120 caramel cookies.

It is also mentioned that the expected demand for the chocolate chip cookies is at least 120 and the bakery can produce no more than 140 cookies. To be profitable the company must sell at least 240 cookies daily. These can be written mathematically as follows:

$$Max\ P = 0.88x + 0.75y$$

$$\begin{cases} x + y \geq 240 \\ 120 \geq y \geq 80 \\ 140 \geq x \geq 120 \end{cases}$$

**Step 4** - Choose the method for solving the linear programming problem, we will find the solution of the above problem graphically.

**Step 5** - Construct the graph $P = 0.88x + 0.75y$

**Step 6** - Identify the feasible region, the green area of the graph is the feasibility region.

**Step 7** - Find the Optimum point

i) Now, we must test the vertices of the feasibility region to determine the optimal solution. The vertices are: (120, 120) , (100, 140), (120, 140)

$$For\ po\text{int}\,(120,\ 120)\ \ and\ P = 0.88\ (120) + 0.75\ (120) = \$\ 195.6$$
$$For\ po\text{int}\,(100,\ 140)\ \ and\ P = 0.88\ (100) + 0.75\ (140) = \$\ 193$$
$$For\ po\text{int}\,(120,\ 140)\ \ and\ P = 0.88\ (120) + 0.75\ (140) = \$\ 210.6$$

Basing on these results, the bakery should manufacture 120 caramel cookies and 140 chocolate cookies daily to maximize the profit. Now, we will proceed to solve the part b of the problem. The maximum profit that can be generated in a day is $210.6.

**Answer 3:**

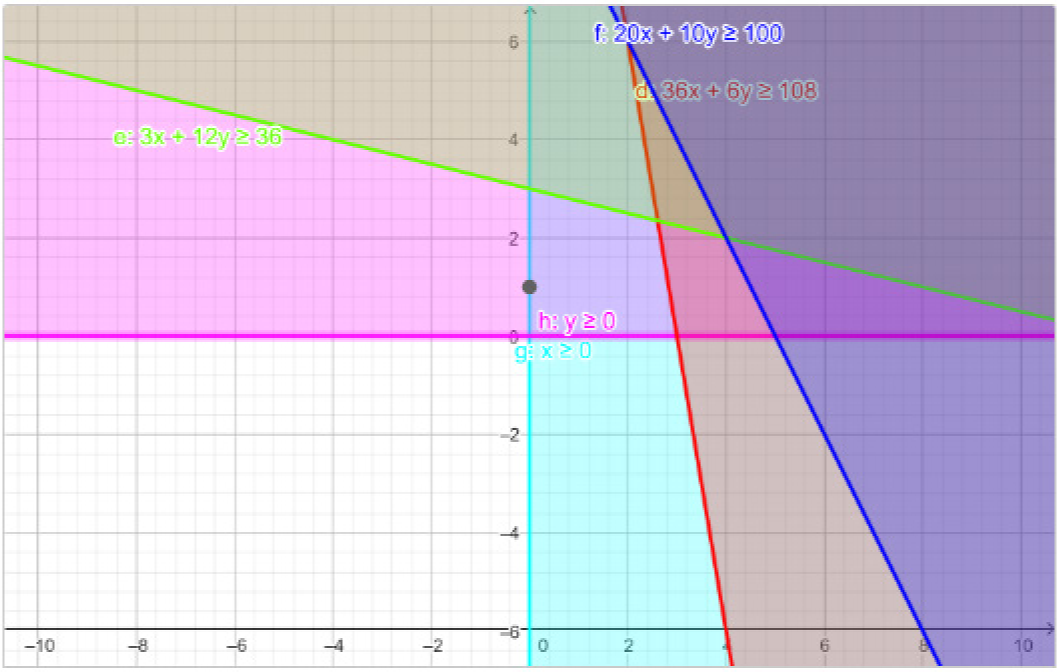
Formulation of a linear Programming problem to minimize the total cost, using graphical method you have to summarize the data of the given problem in the following table:

| Nutrient constituents | Nutrient content in product | | Minimum amount of nutrient |
|---|---|---|---|
| | A | B | |
| X<br>Y<br>Z | 36<br>3<br>20 | 6<br>12<br>10 | 108<br>36<br>100 |
| Cost of product | F20 | F40 | |

Let $u$ be the nutrient content in product A, and $v$ be the nutrient content in product B. The mathematical model can be written as:

$$Minimize\ C = 20u + 40v$$

$$st.\begin{cases} 36u + 6v \geq 108 \\ 3u + 12v \geq 36 \qquad u, v \geq 0 \\ 20u + 10v \geq 100 \end{cases}$$




The minimum point will be (4,2) with the minimum cost of 160F

## 2.7. Summary of the unit

When using LP to solve an organizational problem, the seven-step procedure outlined below should be followed:

**Step 1: Formulate the Problem:** LP analyst first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

**Step 2: Observe the System:** Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step 4) a mathematical model of the organization's problem.

**Step 3: Formulate a Mathematical Model of the Problem:** The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

**Step 4: Verify the Model and Use the Model for Prediction:** The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

**Step 5: Select a Suitable Alternative**: Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives. Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

**Step 6: Present the Results and Conclusions of the Study:** In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the decision maker(s) choose the one that best meets her/his/their needs. Presenting the results of LP study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or 3.

**Step 7:Implement and Evaluate Recommendation:** If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.

**We summarize that t**he Maximization Linear Programming Problems

- Write the objective function,
- Write the constraints,
- For the standard maximization linear programming problems, constraints are of the form: $ax+by\leq c$,
- Since the variables are non-negative, we include the constraints: $x\geq 0$ ; $y\geq 0$
- Graph the constraints.
- Shade the feasibility region.
- Find the corner points.

**Linear Programming:** It is an optimization method for a linear objective function and a system of linear inequalities or equations. The linear inequalities or equations are known as constraints. The quantity which needs to be maximized or minimized (optimized) is reflected by the objective function. The fundamental objective of the linear programming model is to look for the values of the variables that optimize (maximize or minimize) the objective function.

We know that in linear programming, we subject linear functions to multiple constraints. These constraints can be written in the form of linear inequality or linear equations. This method plays a fundamental role in finding optimal resource utilization. The word "linear" in linear programming depicts the relationship between different variables. It means that the variables have a linear relationship between them. The word "programming" in linear programming shows that the optimal solution is selected from different alternatives.

We assume the following things while solving the **linear programming problems**:

- The constraints are expressed in the quantitative values. These are the limitations set on the main objective function. These limitations must be represented in the mathematical form.
- There is a linear relationship between the objective function and the constraints. This function is expressed as a linear function and it describes the quantity that needs optimization.
- The objective function which is also a linear function needs optimization. There is a linear relationship between the variables of the function.
- **Non-negativity:** The value of the variable should be zero or non-negative.

## 2.8. Additional Information for Teachers

- The most crucial step in addressing a linear programming issue is formulating it using the provided data. The following are the steps for solving linear programming problems:
- Determine the choice factors
- Develop the objective function
- Determine whether the function should be decreased or maximized
- Record the limitations
- Verify that decision variables are either larger than or equal to 0. (Non-negative inhibition)
- Utilize either the simplex or graphical method to resolve the linear programming issues.

**1. Traits of a linear programming task**

- A problem being solved through linear programming will have the following traits or characteristics:
- Subject to constraints: Regarding the resource, one should represent the restrictions in mathematical form.
- Geared towards an objective function: The objective function of an issue should be described quantitatively.
- A linear relationship: The function's connection across two or more independent variables must be linear. It indicates that the variable's degree is one.
- Includes only finite numbers: There should be output and input numbers that are both finite and infinite. The optimum solution is not implementable if the function contains an unlimited number of elements.
- Does not include negative values: The variable's value must be zero or positive. The value should not be negative.
- Hinges on decision variables: The result is determined by the decision variable. It provides the final solution to the issue. The first step in solving any issue is to determine the decision factors.
- TORA, LINGO, etc. also can be used to solve linear programming problems.
- In this course we have seen graphical method but there are several approaches to solving linear programming problems.

2. **Solving linear programming problems using R (Using Laptop, and software)**

- Linear programming is an excellent tool for decision-making optimization. Several R programs, such as the lpSolve R package, enable the solution of linear programming difficulties. lpSolve is an R extension that allows links to a C-based framework for linear programming problem-solving. With only a few bits of open-source code, you may get statistically significant information (sensitivity analysis).
- Whereas, other free optimization solutions are available, having a linear programming R code in one's individual code repository may save a considerable amount of time by eliminating the need to start writing the formula from scratch and requiring only the modification of the coefficients and signs of the respective matrices. This is helpful since R is regularly used for data science and statistical analysis.

3. **Graphical linear programming**

The technique of resolving a linear equation system by generating a graph is often referred to as the graphical method. The same holds true for linear programming issues. Using graphical approaches, it is simple to solve any optimization programming issue with just two variables. These variables may be referred to as $x_1$ and $x_2$, and most of the analysis can be performed on a two-dimensional graph using these variables. The graphical approach for solving linear programming employs the extreme or corner point's method and the iso-profit (cost) efficiency line method.

**Examples of Linear Programming**

The Linear Programming Problem (LPP) involves finding the best value of a given linear function. The ideal value may either be the largest or smallest one. Here, the linear function provided is regarded as an objective function. The objective function may include several variables dependent on the situation and must fulfill the linear constraints, a collection of linear inequalities. One may utilize linear programming issues to determine the ideal solution for manufacturing, diet, transportation, and allocation problems, among others. Listed below are a few illustrations of the kind of issues commonly addressed by linear programming techniques:

**Example 1: Optimizing dietary needs and cost constraints**

A doctor wants to combine two food kinds such that the mixture's vitamin content includes a minimum of 8 elements of vitamin A and ten elements of vitamin C. Food 'I' includes 2 vitamin A units per kilogram and 1 vitamin C unit per kilogram. Food ‘II’ has 1 vitamin A unit per kilogram and 2 vitamin C units per kilogram. Food ‘I’ is priced at $5 per kilogram, whereas Food ‘II’ costs $7 per kilogram. To minimize the price of such a combination, this may be expressed as a problem of linear programming.

**Example 2: Optimizing food ingredients and food volume**

One type of cake calls for 200g of flour and 25g of fat, but another one calls for 100g of flour and 50g of fat. This issue may be expressed as a linear programming problem to determine the highest proportion of cake that can be baked using 5kg of wheat and 1kg of fat. It also implies that there are sufficient quantities of the other cake-making components.

**Example 3: Optimizing goods transportation costs**

Consider a manufacturing business with two plants in cities F1 & F2 and three retail outlets in cities C1, C2, or C3. Monthly demand at retail locations is 8, 5, and 2 units, whereas monthly supply at manufacturers is 6 and 9, accordingly. Notice that the entire supply and demand are equal. We are also provided with the cost of transporting one unit from manufacturing to retail outlets. This linear programming challenge aims to estimate the amount that must be shipped from each manufacturer to each retail area to satisfy demand at the lowest possible total shipping cost.

**Example 4: Optimizing product sales to arrive at maximum profit**

**A bakery produces two types of cookies:** chocolate chip and caramel. The bakery anticipates daily demand for a minimum of 80 caramelized & 120 chocolate chip cookies. Due to a lack of raw materials and labor, the bakery can produce 120 caramel cookies and 140 chocolate chip cookies daily. For the bakery to be viable, it must sell a minimum of 240 cookies each day. Every chocolate chip cookie served generates $0.75 in profit, whereas each caramel biscuit generates $0.88. The solution to the number of chocolate chip and caramel cookies that the bakery must produce each day to maximize profit may be determined using linear programming.

**Note:** Emphasize on how to write a mathematical model considering two variables only, we always set condition/restrictions.

## 2.9. Additional activities

### 2.9.1. Remedial activities.

1. An advertising company wishes to plan its advertising strategy in three different media- television, radio and magazines. The purpose

of advertising is to reach as large a number of potential customers as possible. Following data has been obtained from market survey:

| All in RWF | Television | Radio | Magazine I | Magazine II |
|---|---|---|---|---|
| Cost of an advertising unit | 30,000 | 20,000 | 15,000 | 10,000 |
| No of potential customers per unit | 200,000 | 600,000 | 150,000 | 100,000 |
| No of female customers per unit | 150,000 | 400,000 | 70,000 | 50,000 |

The company wants to spend no more than RWF 450,000 on advertising. Following are that further requirements that must be met:

i) At least 1 million exposures take place among female customers,'

ii) Advertising on magazines be limited to RWF 150,000,

iii) At least 3 advertising units be bought on magazine 1 and 2 units on magazine II,

iv) The number of advertising units on television and radio should each be between 5 and 10. Formulate an L.P model for the problem.

**Formulation of linear Programming Model**

**Step 1:** The key decision to be made is to determine the number of advertising units in television, radio, magazine I and magazine II.

**Step 2:** Let $x_1, x_2, x_3, x_4$ represent the number of these advertising units in television, radio, magazine I and magazine II.

**Step 3:** Feasible alternatives are set of values $x_1, x_2, x_3, x_4$ where $x_1, x_2, x_3, x_4$ all are $\geq 0$

**Step 4:** The objective is to maximize the total number of potential customers.

i.e., Maximize $z = 10^5 \left(2x_1 + 6x_2 + 1.5x_3 + x_4\right)$

**Step5:** Constraints are on the advertising budget:

$$30,000x_1 + 20,000x_2 + 15,000x_3 + 10,000x_4 \leq 450,000$$

Or

$$30x_1 + 20x_2 + 15x_3 + 10x_4 \le 450$$

On number of female customers reached by the advertising company:

$$150,000x_1 + 400,000x_2 + 70,000x_3 + 50,000x_4 \ge 1,000,000$$

Or

$$15x_1 + 40x_2 + 7x_3 + 5x_4 \ge 100$$

On expenses on magazine advertising:

$$15,000x_3 + 10,000x_4 \le 150,000$$

Or

$$15x_3 + 10x_4 \le 150$$

On number of units on magazines: $x_3 \ge 3; x_4 \ge 2$

On number of units on television: $5 \le x_1 \le 10$

On number of units on radio: $5 \le x_1 \le 10$

### 2.9.2. Consolidation activities:

1. A firm produces three products. These products are processed on three different machines. The time required manufacturing one unit of each of the three products and the daily capacity of the three machines are given in the table below.

| Machine | Time per unit (minutes) | | | Machine capacity ( minutes/day) |
|---|---|---|---|---|
| | Product 1 | Product 2 | Product 3 | |
| M1 | 2 | 3 | 2 | 440 |
| M2 | 4 | - | 3 | 470 |
| M3 | 2 | 5 | - | 430 |

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product 1, 2 and 3 is \$4, \$3 and \$6 respectively. It is assumed that all the amounts produced are consumed in the market.

Formulation of Linear programming Model

**Step 1:**

From the study of the situation find the key-decision to be made. It is this connection, looking for variables helps considerably. In the given situation key decision is to decide the extent of product 1, 2 and 3, as the extents are permitted to vary.

**Step 2:**

Assume symbols for variable quantities noticed in step 1. Let the extents (amounts) of products, 1, 2 and 3 manufactured daily be $x_1, x_2, and\ x_3$ respectively.

**Step 3:**

Express the feasible alternatives mathematically in terms of variables. Feasible alternatives are those which are physically, economically and financially possible. In the given situation feasible alternatives are sets of values of $x_1, x_2$ and $x_3$, where $x_1, x_2, x_3 \geq 0$

Since negative production has no meaning and is not feasible.

**Step 4:**

Mention the objective quantitatively and express it as a linear function of variables. In the present situation, objective is to maximize the profit. i.e., maximize $z = 4x_1 + 3x_2 + 6x_3$

**Step 5:**

Put into words the influencing factors or constraints. These occur generally because of constraints on availability (resources) or requirements (demands). Express these constraints also as linear equalities/inequalities in terms of variables. Here, constraints are on the machine capacities and can be mathematically expressed as

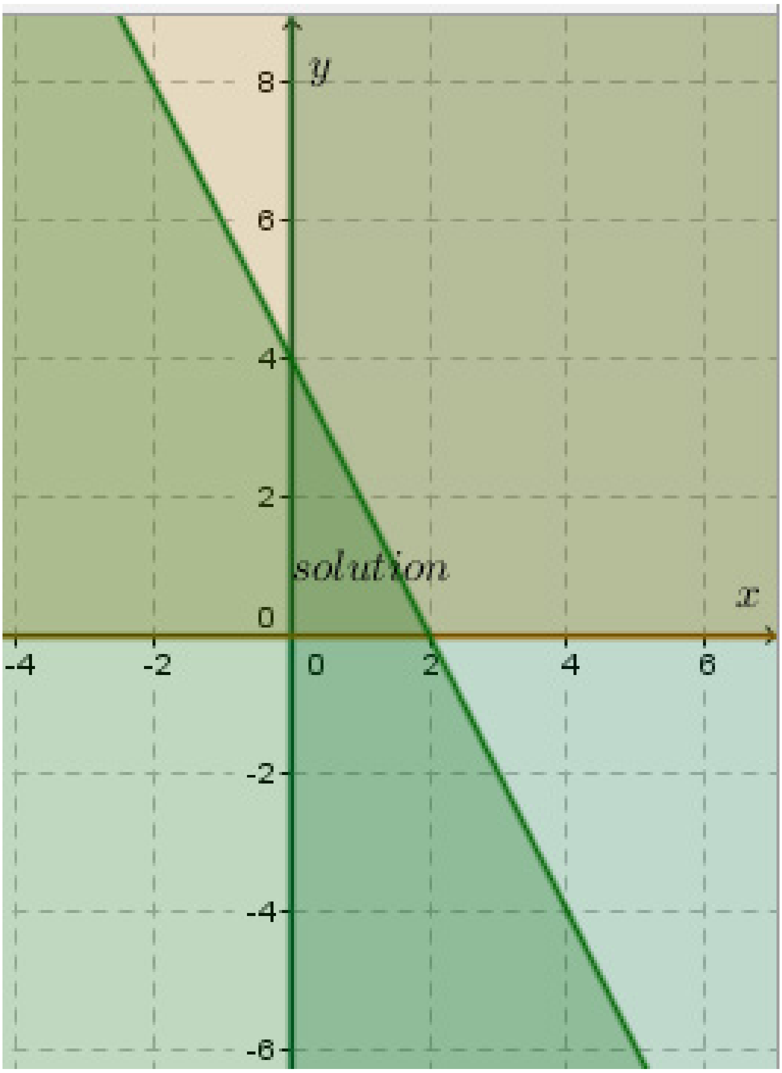
$$2x_1 + 3x_2 + 2x_3 \leq 440$$
$$4x_1 + 0x_2 + 3x_3 \leq 470$$
$$2x_1 + 5x_2 + 0x_3 \leq 430$$

**Extended activities:**

Solve graphically the following system:

c. $\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x = y \leq 4 \end{cases}$

**Solution:**



# UNIT 3 | INTEGRALS

## 3.1 Key unit competence

Use integration to solve mathematical and financial related problems involving marginal cost, revenues and profits, elasticity of demand, and supply

## 3.2 Prerequisite

The students will perform well in this unit if they have a good background on

- **Differentiations/derivatives** (unit 2 from senior 5)
- **Basic concepts of Algebra** (Unit1 from senior 4)
- **Polynomial functions, equations and inequalities** (unit 2 from senior 4)

## 3.3 Cross-cutting issues to be addressed

- Inclusive education (promoting education for all in teaching and learning process)
- Peace and value Education (respect the views and thoughts of others during class discussions)
- Gender: Provide equal opportunity for boys and girls to participate in class

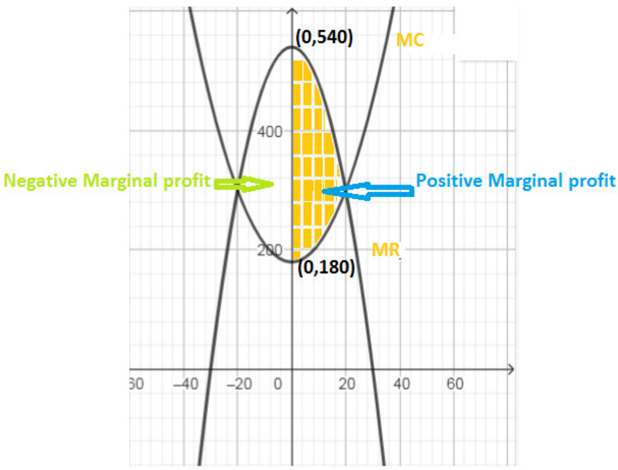
## 3.4 Guidance on introductory activity 3

- Invite students to work in small group, discuss and find out the answers for the introductory activity 3 from the student's book
- Facilitate students' discussions and ask them to avoid noise or other unnecessary conversations.
- During discussions, let students think of different ways to solve the given problem
- Walk around in all groups to provide assistance if necessary.

- Invite group members to present their findings and encourage boys and girls to actively participate in presentations.
- Basing on students' experience, prior knowledge and abilities shown in answering the questions for this introductory activity, use different probing questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

### Answer for introductory activity 3

a) Graph




b) The positive marginal profit and negative marginal profit are shown on graph in (a)

c) Profit is maximized when $MC = MR$. Therefore,

$$180 + 0.3q^2 = 540 + 0.6q^2$$

$$q^2 = 400 \Rightarrow q = 20$$

d) To find actual profit $(\pi)$, we know now integrate to get $TR \ and \ TC$ and then $TC \ from \ TR$

$$TR = \int MRdq = \int \left(540 - 0.6q^2\right)dq = 540q - 0.2q^3$$

$$TC = \int MCdq + TFC = \int \left(180 - 0.3q^2\right)dq + 65 = 180q - 0.1q^3 + 65$$

e) To find actual profit $(\pi) = TR - TC$

$$= 540q - 0.2q^3 - \left(180q + 0.1q^3 + 65\right)$$

$$= 360q - 0.3q^3 - 65$$

Thus, when $q = 20$ the maximum profit level is $360(20) - 0.3(20)^3 - 65 = 4,735$

## 3.5 List of lessons

| Headings | # | Lesson title/sub-headings | Learning objectives | Number of periods |
|---|---|---|---|---|
| 3.1 Indefinite integral | Introductory activity | | Arouse the curiosity of students on the content of unit 3. | 1 |
| | 1 | Definition of indefinite integral | Applydifferentiation rules to generate definition of indefinite integral | 2 |
| | 2 | Properties of indefinite integral | Use properties of integrals to simplify the calculation of integrals. | 2 |
| | 3 | Primitive functions | Extend definition and properties of indefinite integral to primitive functions | 2 |
| 3.2 Techniques of integration | 1 | Separable of variables (Substitution method) | Use indefinite integral properties to integrate function by substitution. | 2 |
| | 2 | Integration by parts | Use indefinite integral properties to integrate function by parts. | 4 |
| | 3 | Decomposition/ Simple fraction | Use indefinite integral properties to integrate function by decomposition/ Simple fraction. | 4 |

| | | | | |
|---|---|---|---|---|
| 3.3 definite integral | 1 | Definition and properties | Extend the concepts of indefinite integrals to definite integrals | 2 |
| | 2 | Techniques of integration of definite integral | Extend the concepts of indefinite integrals to definite integrals to solve problems involving economics and finance | 4 |
| 3 . 4 Application of integration | 1 | calculation of marginal cost, revenues and profits, | Use integration to derive profit functions from the marginal revenue functions | 4 |
| | 2 | Elasticity of demand, and supply | Use integration to calculate elasticity of demand, and supply | 3 |
| | | Present, Future Values of an Income Stream and Growth Rates | Use integration to calculate Present, Future Values of an Income Stream and Growth Rates | 3 |
| **3.5 End unit assessment** | | | | **3** |

## Lesson 1: Definition of indefinite integral

**a) Learning objective:**

Use indefinite properties to evaluate indefinite integral

**b) Teaching resources:**

Student's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

**c) Prerequisites/Revision/Introduction:**

Students will perform well in this lesson if they are skilled enough in the content of differentiation learnt in senior five

**d) Learning activities**

- Invite Students to work in group and do the learning activity 3.1.1 form the S6 Mathematics student's book;
- Move around for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills on definition of integral and the integral representation.
- Ask students to do the application activity 3.1.1 and evaluate whether lesson objectives were achieved to assess their competences.

**Answers for learning activity 3.1.1**

i) $F(x)=x^2+c$ where c is a constant, since $F'(x)=(x^2+c)'=2x$

ii) There are infinitely many possibilities for $F(x)$ because c can take different values in the set of real numbers.

iii) They all differ by a constant c.

**e) Answers for the application activity 3.1.1**

a) $\int 2x dx = x^2 + c$

b) $\int 3e^x dx = 3e^x + c$

c) $I = \int x^2 dx = \frac{1}{3}x^3 + c$

d) $\int (3x^3+1) dx = \int 3x^3 dx + \int dx = \frac{3}{4}x^4 + x + c$

## Lesson 2: Properties of indefinite integral

### a) Learning objectives:

Use properties of integrals to simplify the calculation of integrals.

### b) Teaching resources:

Student's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

### c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good background on concepts of Natural, rational, integers, irrational and real numbers learnt in previous lesson or learnt in ordinary level.

### d) Learning activities

- Invite students to work in small groups and do the learning activity 3.1.2 in their Mathematics books;
- Move around in the class for facilitating students where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a teacher, harmonize findings from students' presentation;
- Use different probing questions and guide students to explore the content and examples given in the students' book and lead them to discover properties of indefinite integral
- Invite students to apply these properties in evaluating indefinite integrals
- After this step, guide students to do the application activity 3.1.2 and evaluate whether lesson objectives were achieved.

**Answers of learning activity 3.1.2.**

a) i) $I_1 = \int f(x)\,dx = \int 5dx = 5x + c$

ii) $I_2 = \int g(x)\,dx = \int \frac{dx}{x} = \ln|x| + c$

b) $I=\int (f+g)(x)\,dx=\int \left(5+\frac{1}{x}\right)dx=\int f(x)dx+\int g(x)dx=5x+\ln|x|+c$

$I=I_1+I_2=5x+\ln|x|+c$ , yes there are the same.

c) $I=\int (f-g)(x)\,dx=\int \left(5-\frac{1}{x}\right)dx=\int f(x)dx+\int g(x)dx=5x-\ln|x|+c$

Yes, they are the same

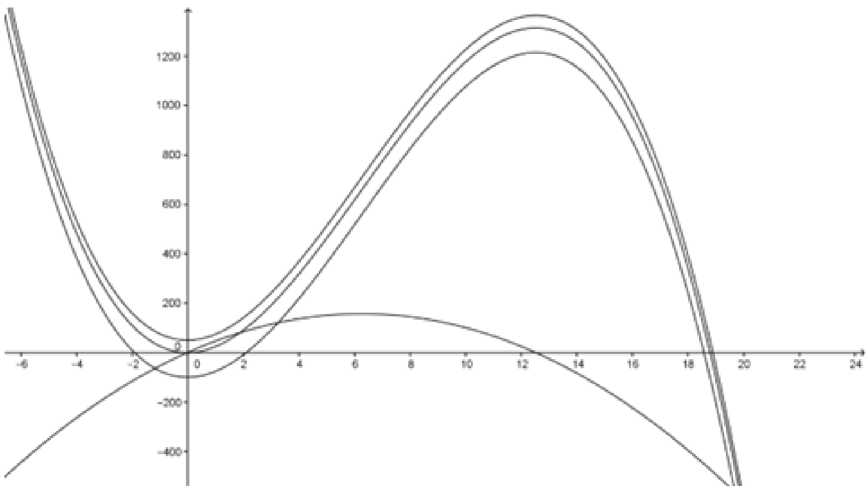
**e) Answers of application activity 3.1.2**

**1. a.** $\int \left(x^3+3\sqrt{x}-7\right)dx=\frac{1}{4}x^4+2\sqrt{x^3}-7x+c$

b. $\int \left(4x-12x^2+8x^6-9\right)dx=2x^2-4x^3+4x^2-9x$

2. a. The given $M(x)=1+50x-4x^2$, Total cost $(TC)=x+25x^2-\frac{4}{3}x^3+c$

b. graph of the marginal cost and three of its corresponding possible total costs



### Lesson 3: Primitive functions

**a) Learning objective:**

Extend definition and properties of indefinite integral to primitive functions

**b) Teaching resources:**

Students's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

**c) Prerequisites/Revision/Introduction:**

Students will learn better this lesson if they have a good background on concept of definition and properties of indefinite integral learnt in previous lesson

Students will learn better this lesson if they have a good background on differentiation learnt in senior five

**d) Learning activities**

- Ask the students to perform the learning activity 3.1.3 in pairs
- Move around in the class to verify students' progress over the work.
- Have groups with different activities to present their answers to the whole class.
- As a teacher, harmonize students' insights by insisting on reversing the direction of formulas for derivatives of elementary functions
- Ask students in pairs, to discuss the primitive functions of elementary functions
- Invite students to use formulas for derivatives of elementary functions
- Use different probing questions and guide students to explore examples and content from the student book to solve related activity
- After this step, guide the students to complete application activity 3.1.3. conduct, assess students' competences and whether the objectives have been achieved.

**Answers of learning activity 3.1.3.**

a) i. $F(x)=\ln(x) \to F'(x)=\left(\ln(x)\right)' = \frac{1}{x}$

ii. $\int \frac{1}{x}dx = \ln|x| + c$

b) The functions $F(x)$ and $f(x)$ has relationship because $F'(x)=f(x)$

### e) Answers of application activity 3.1.3

a) $\int e^{3x+1}dx = \frac{1}{3}e^{3x+1} + c$

b) $\int 3^x dx = \frac{3^x}{\ln 3} + c$

c) $\int (8 - x^5) dx = 8x - \frac{1}{6}x^6 + c$

## Lesson 4: Integration by substitution or change of variable

### a) Learning objective:

Use indefinite integral properties to integrate function by substitution. Method

### b) Teaching resources:

students's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

### c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good background on concept of definition and properties of indefinite integral learnt in previous lesson.

Students will learn better this lesson if they have a good background on differentiation learnt in senior five (Unit 2).

### d) Learning activities

- Invite students to work in pairs and perform the learning activity 3.2.1 in their Mathematics books;
- Move around in the class for facilitating students in their respective pairs where necessary;
- Verify and identify pairs with different working steps;
- Invite one member from each pairs with different working steps to present their work where they must explain the working steps;
- As a teacher, harmonize their findings from the presentation.;

- Use different probing questions and guide students to explore the content and examples given in the students' book and lead them to discover how to evaluate integration by substitution method
- After this step, guide students to do the application activity 3.2.1 and evaluate whether lesson objectives were achieved.

**Answers of learning activity 3.2.1**

Given $I=\int e^{5x+2}dx$ ; let $u=5x+2 \Rightarrow \frac{du}{dx}=5 \Rightarrow dx=\frac{1}{5}du$

$$I=\frac{1}{5}\int e^{u}du=\frac{1}{5}e^{u}+c$$

Therefore, $\int e^{5x+2}dx=\frac{1}{5}e^{5x+2}+c$

**e) Answers for application activity 3.2.1**

1. $\int (2x+1)^4 dx$, let $t=2x+1$

$dt=2xdx \rightarrow dx=\frac{dt}{2}$

$\frac{1}{2}\int (t)^4 dt=\frac{1}{10}t^5+c \rightarrow \frac{1}{10}(2x+1)^5+c$

$2.a.\quad \int (x^2+1)2xdx$

Let $u=x^2+1 \Rightarrow \frac{du}{dx}=2x \Rightarrow 2xdx=du$

$\Rightarrow \int udu=\frac{1}{2}u^2+c \Rightarrow \frac{1}{2}(x^2+1)^2+c$

$b.\quad \int x^2 e^{x^3}dx$

$=\frac{1}{3}e^{x^3}+c$

$c.\quad \int (2x+1)e^{x^2+x+2}dx=e^{x^2+x+2}+c$

## Lesson 5. Integration by Parts

### a) Learning objective:

Use indefinite integral properties to integrate function by parts.

### b) Teaching resources:

Student’s book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with mathematical software such as Geogebra, Microsoft Excel, Math-lab or graph-application and internet can be used...

### c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good background on concept of definition and properties of indefinite integral learnt in previous lesson

Students will learn better this lesson if they have a good background on differentiation learnt in senior five and perquisites on previous lesson of this unit.

### d) Learning activities

- Invite students to be organized in small groups and provide clear instructions about learning activity 3.2.2.
- Invite students to use their student’s book and perform learning activity 3.2.2
- Switch to each group to ensure all students are actively participating
- Move around in the class for facilitating students where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As teacher, harmonize their answers from presentation;
- Use different probing questions and guide students to explore the content and examples given in the student’ s book and lead them to discover how to use integration by part to evaluate given integrals.
- After this step, guide students to do the application activity 3.2.2 and evaluate whether lesson objectives were achieved.

## Answers of learning activity 3.2.2

1. $\frac{d}{dx}f(x)=e^x\frac{d}{dx}(x-1)+(x-1)\frac{d}{dx}(e^x)$

$\Rightarrow e^x+xe^x-e^x=xe^x$

$\frac{d}{dx}f(x)=xe^x$

2. From (1), $\int xe^x dx=(x-1)e^x+c$

3. Let us consider $f(x)=xe^x$ from (1) to verify $\int uvdx=\int udx\int vdx$,

If $u=x, v=e^x$ , thus $\int uvdx=\int xe^x dx$, while $\int udx\int vdx=\int xdx\int e^x dx$

From (2), we have: $\int xe^x dx=(x-1)e^x+c$

Let us find $\int udx\int vdx$:

$\int udx\int vdx=\int xdx\int e^x dx=\frac{1}{2}x^2\times e^x+c$

Therefore, $\int uvdx\neq\int udx\int vdx$

Finally, $\int udv=uv-\int vdu$

## e) Answers for application activity 3.2.2

a) $\int x^3\ln xdx$

Since $\ln x$ is a logarithmic function and $x^3$ is an algebraic function, let

$u=\ln x,\ dv=x^3dx\rightarrow du=\frac{1}{x}dx,\ v=\int x^3dx=\frac{x^4}{4}$

$\int x^3\ln x=uv-\int vdu$

$=(\ln x)\frac{x^4}{4}-\int\frac{x^4}{4}\frac{1}{x}dx=\frac{x^4}{4}(\ln x)-\frac{x^4}{16}+c$

b) $I=\int xe^{3x}dx \Rightarrow I=\frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}+c$

c) Given $\int xe^{-2x}dx$

Let $u=x \Rightarrow du=dx$ and $dv=e^{-2x}dx \Rightarrow v=-\frac{e^{-2x}}{2}$, then,

$\int xe^{-2x}dx=-\frac{xe^{-2x}}{2}+\frac{1}{2}\underbrace{\int e^{-2x}dx}_{I_1}$ For $I_1=\int e^{-2x}dx=-\frac{1}{2}e^{-2x}$ Thus,

$$\int xe^{-2x}dx=-\frac{xe^{-2x}}{2}-\frac{1}{4}e^{-2x}=-\frac{x}{2e^{2x}}-\frac{1}{4e^{2x}}+c, c\in\mathbb{R}$$

## Lesson 6. Integration by Decomposition/ Simple fraction

### a) Learning objective:

Use indefinite integral properties to integrate function by decomposition/ Simple fraction

### b) Teaching resources:

Manila paper, Markers, Calculators, student's book and other Reference textbooks to facilitate research, Mathematical models and Internet connection where applicable

### c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good background on concept of definition and properties of indefinite integral and perquisites on previous lesson of this unit.

Students will learn better this lesson if they have a good background on differentiation learnt in senior five

### d) Learning activities

- Organize the students into small groups;
- Provide clear instructions and introduce the learning activity 3.2.3 by guiding students in this activity.
- Ask students to use the student's book and discuss on the learning activity 3.2.3

- Switch to each group to ensure all students are actively participating
- Call upon groups to present their findings and harmonize their answers;
- Use various probing questions to guide students to explore examples and content related to integration by Decomposition/ Simple fraction
- Ask students to give examples of where integration by Decomposition/ Simple fraction can be used
- After this step, guide the students to do the application activity 3.2.3, assessing their competences and evaluating whether the lesson objectives have been met.

## Answers of learning activity 3.2.3

*a.* $\frac{x-2}{x^2+2x}=\frac{x-2}{x(x+2)}\Rightarrow$

$$\frac{A}{x}+\frac{B}{x+2}=\frac{Ax+2A+B}{x(x+2)}$$

$$\Rightarrow\begin{cases}A+B=1\\2A=-2\end{cases}\Rightarrow\begin{cases}A=-1\\B=2\end{cases}$$

$$I=\int\frac{x-2}{x(x+2)}dx=-\int\frac{dx}{x}+2\int\frac{dx}{x+2}$$

$$I=2\ln|x+2|-\ln|x|+c$$

*b.* $\frac{x}{x^2+3x+2}=\frac{x}{(x+1)(x+2)}$

$$\Rightarrow I=\int\frac{x}{(x+1)(x+2)}dx\Rightarrow\frac{A}{x+1}+\frac{B}{x+2}=\frac{x}{(x+1)(x+2)}$$

$$\Rightarrow Ax+2A+Bx+B=x$$

$$\Rightarrow\begin{cases}A+B=1\\2A+B=0\end{cases}\Rightarrow\begin{cases}A=-1\\B=2\end{cases}$$

$$\Rightarrow 2\int\frac{dx}{x+2}-\int\frac{dx}{x+1}\Rightarrow I=2\ln|x+2|-\ln|x+1|+c$$

c. $\dfrac{2}{x^2-4} \Rightarrow \dfrac{2}{(x-2)(x+2)}$

$$\Rightarrow \frac{A}{x-2}+\frac{B}{x+2} \Rightarrow \frac{Ax+2A+Bx-2B}{(x-2)(x+2)}=\frac{2}{(x-2)(x+2)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ 2A-2B=2 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \end{cases}$$

$$\Rightarrow \frac{1}{2}\int\frac{dx}{x-2}-\frac{1}{2}\int\frac{dx}{x+2}$$

$$\Rightarrow \frac{1}{2}\ln|x-2|-\frac{1}{2}\ln|x+2|+c \text{ or } I=\ln\sqrt{\frac{x-2}{x+2}}+c$$

## e) Answers of application activity 3.2.3

1. $I=\int\dfrac{2dx}{x^2-1} \Rightarrow I=\ln\left|\dfrac{x-1}{x+1}\right|+c$

2. $I=\int\dfrac{xdx}{x^2+3x+2} \Rightarrow I=2\ln|x+2|-\ln|x+1|+c$

3. $given \int\dfrac{x+3}{x^2-5x+4}dx$

We need to factorize: $x^2-5x+4=(x-4)(x-1)$

$$\frac{x+3}{x^2-5x+4}=\frac{x+3}{(x-4)(x-1)}=\frac{A}{x-4}+\frac{B}{x-1} \rightarrow x+3=A(x-1)+B(x-4)$$

For $x=1, B=-\dfrac{4}{3}$ and for $x=4,\ A=\dfrac{7}{3}$

Therefore, $\int\dfrac{x+3}{x^2-5x+4}dx=\int\dfrac{Adx}{x-4}+\int\dfrac{Bdx}{x-1} \rightarrow \dfrac{7}{3}\int\dfrac{dx}{x-4}-\dfrac{4}{3}\int\dfrac{dx}{x-1}$

$$\frac{7}{3}\ln|x-4|-\frac{4}{3}\ln|x-1|+c$$

$\Rightarrow \frac{7}{3}\ln|x-4|-\frac{4}{3}\ln|x-1|+c$ or, the same as $\ln\sqrt[3]{\frac{(x-4)^7}{(x-1)^4}}+c$

## Lesson 7. Definition and properties of definite integrals

### a) Learning objective:

Extend the concepts of indefinite integrals to definite integrals

### b) Teaching resources:

Digital materials including calculator, Manila paper, Markers, Calculators, student's book and other reference textbooks to facilitate research, Mathematical models and Internet connection where applicable.

### c) Prerequisites/Revision/Introduction:

Students will learn better this lesson if they have a good background on concept of definition and properties of indefinite integral and perquisites on previous lesson of this unit.

Students will learn better this lesson if they have a good background on differentiation learnt in senior five

### d) Learning activities

- In small group discussions, invite students to perform learning activity 3.3.1 in student's book
- Use gallery walk, students share their answers to others by rotating and ask support on challenging points they faced in their group.
- Move around to see student's progress in their respective groups and give guidance.
- Invite groups with different working steps to present their answers then, harmonize the presented findings.
- After doing this step, use different probing questions and guide students to discover definition and properties of definite integrals
- Guide students to explore and perform examples on properties and definition of definite integral found in content summary.
- Invite students to ask questions and share their best practices to their classmates

- After this, invite students to do application activity 3.3.1, assessing their competences and evaluating whether the lesson objectives have been met.

**Answers for learning Activity 3.3.1**

1. a. $\int_1^2 f(x)dx = \int_1^2 (x^3+3)dx = \left[\frac{x^4}{4}+3x\right]_1^2 = \frac{27}{4}$;

b. $\int_1^2 g(x)dx = \int_1^2 (-2x^2)dx = \left[-\frac{2x^3}{3}\right]_1^2 = -\frac{14}{3}$;

c. $\int_1^2 (f+g)(x)dx = \int_1^2 (x^3+3-2x^2)dx = \left[\frac{x^4}{4}+3x-\frac{2x^3}{3}\right]_1^2 = \frac{25}{12}$

2. $\int_1^2 (f+g)(x)dx$, $\int_1^2 f(x)dx + \int_1^2 g(x)dx = \frac{27}{4} + \left(-\frac{14}{3}\right) = \frac{25}{12}$

Therefore, $\int_1^2 (f+g)(x)dx$ and $\int_1^2 f(x)dx + \int_1^2 g(x)dx$ are the same since they have the same answer

3. $from(2), \int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$,

$where, \int_a^b f(x)dx = F(b)-F(a)\ and \int_a^b g(x)dx = G(b)-G(a)$

## e) Answers of application activity 3.3.1

1. $I = \int_0^3 xdx = \left[\frac{x^2}{2}\right]_0^3 = \frac{9}{2}$

2. $I = \int_1^2 (x^2-x)dx = \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_1^2 = \frac{5}{6}$

3. $I = \int_1^2 (3x^2-6x)dx = \left[\frac{3x^3}{3}-\frac{6x^2}{2}\right]_1^2 = -2$

$$I = \int_{-1}^2 (x^3+3x^2-4)dx = \left[\left(\frac{x^4}{4}+\frac{3x^3}{3}-4x\right)\right]_{-1}^2 = \frac{3}{4}$$

## Lesson 8. Techniques of integration of definite integral

### a) Learning objective:

Extend the concepts of indefinite integrals to definite integrals to solve problems involving economics and finance

### b) Teaching resources:

Digital materials including calculator, Manila paper, Markers, Calculators, student's book and other reference textbooks to facilitate research, Mathematical models and Internet connection where applicable.

### c) Prerequisites/Revision/Introduction:

Students will do well in this lesson if they have enough skills on differentiation learnt in senior five and indefinite integral learnt in previous lessons

### d) Learning activities

- In group discussions, invite students to perform learning activity 3.3.2 in student's book
- Use gallery walk, students share their answers to others by rotating and ask support on challenging points they faced in their group.
- Teacher moves around to see student's progress in their respective groups.
- Invite groups with different working steps to present their answers then, harmonize the presented answers.
- After doing this step, use different questions and guide students to discover techniques can be used to evaluate definite integrals
- Invite students to read from content summary in student book. With clear examples, insist on techniques used to evaluate definite integrals
- Guide students to deal with techniques of integration of definite integral through examples
- After this, invite students to perform application activity 3.3.2, assessing their competences and evaluating whether the lesson objectives have been met.

**Answers for learning activity 3.3.2**

1. $f(x) = e^{x^2}$

   i) Let $t = x^2$, then $x = 0 \Rightarrow t = 0 \; and \; x = 2 \Rightarrow t = 4$

ii) $t = x^2$, then $dt = 2xdx, \to dx = \frac{dt}{2x}$

iii) $\int_0^2 2xe^{x^2}dx \Rightarrow \int_0^4 2xe^t \frac{dt}{2x} = \int_0^4 e^t dt = e^4\Big|_0^4 = e^4 - 1$

iv) It is clear that when we apply the substitution method, we also substitute boundaries to keep integral the same.

### e) Answers for application activity 3.3.2

1. $\int_0^1 \ln(1+x)dx$ using integration by parts

Let $u = \ln(1+x) \Rightarrow du = \frac{1}{1+x}$ *and* $dv = dx\ v = x + c$

$I = \int_q^b udv = uv - \int_a^b vdu$ becomes

$$I = \int_0^1 \ln(1+x) = [x\ln(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}dx$$

$$= \ln 2 - \int_0^1 \left(1 - \frac{x}{1+x}\right)dx = \ln 2 - [x - \ln(1+x)]_0^1$$

$$= -1 + 2\ln 2 = -1 + \ln 4$$

2. Given $\int_1^5 \frac{x+7}{x^3+2x^2}dx$

$$\frac{x+7}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$$x + 7 = A(x^2 + 2x) + B(x+2) + cx^2$$

$$0x^2 + x + 7 = (A+C)x^2 + (2A+B)x + 2B$$

We form three equations to solve for numerators

$0 = A + C$

$1 = 2A + B$

$7 = 2B \rightarrow B = \frac{7}{2}$

Replace $B = \frac{7}{2}$ then $A = -\frac{5}{4}$ and $C = \frac{5}{4}$

$$\int_1^5 \frac{x+7}{x^3+2x^2}dx = \int_1^5 \left(-\frac{5}{4x} + \frac{7}{2x^2} + \frac{5}{4(x+2)}\right)dx$$

$$= -\int_1^5 \frac{5}{4x}dx + \int_1^5 \frac{7}{2x^2}dx + \int_1^5 \frac{5}{4(x+2)}dx$$

$$= -\frac{5}{4}\ln|x|\Big|_1^5 - \frac{7}{2x}\Big|_1^5 + \frac{5}{4}\ln|x+2|\Big|_1^5$$

$$= -\frac{5}{4}(\ln(5) - \ln(1)) - \frac{7}{2}(\frac{1}{5} - 1) + \frac{5}{4}(\ln(5+2) - \ln(1+2))$$

$$= -\frac{5}{4}\ln 5 + \frac{28}{10} + \frac{5}{4}\ln 7 - \frac{5}{4}\ln 3$$

Therefore, $\int_1^5 \frac{x+7}{x^3+2x^2}dx == -\frac{5}{4}\ln 5 + \frac{28}{10} + \frac{5}{4}\ln 7 - \frac{5}{4}\ln 3$

## Lesson 9. Calculation of marginal cost, revenues and profits

### a) Learning objective:

Use integration to derive profit functions from the marginal revenue functions.

### b) Teaching resources:

Digital materials including calculator, Manila paper, Markers, Calculators, student's book and other reference textbooks to facilitate research, Mathematical models and Internet connection where applicable.

### c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they are enough skilled in previous covered lesson of this unit. In addition, they will perform well if they have enough skills on differentiation, indefinite integrals and definite integrals

**d) Learning activities**

- Organize students in small group discussions
- Invite students to work in small group discussion and perform the learning activity 3.4.1 from the student' book;
- As students are working on the given activity, move around to each group for guidance
- Ask the groups to share their answers with other groups and allow them to share the challenging points they faced in their groups.
- As Teacher, try to harmonize students' findings;
- Ask students probing questions to discover the application of integrals.
- Guide students to find more in economics, finance, accounting, and real-life examples that can be solved with the intervention of integrals
- After this, invite students to do application activity 3.4.1, assessing their competences and evaluating whether the lesson objectives have been met

**Answers for learning activity 3.4.1**

1. Let $C(q)$ denote the total cost of producing $q$ units. Then the marginal cost is the derivative

$$\frac{dC}{dq}=3(q-4)^2.$$

The increase in cost if production raised from $6$ *units to* $10$ *units* is given by the definite integral:

$$C(q)=\int_6^{10}3(q-4)^2dx=\left[(q-4)^3\right]_6^{10}=208\$.$$

**e) Answers for application activity 3.4.1**

1. The total cost function is given by

$$\int\left(7.5q^2-26q+50\right)dq=2.5q^3-13q^2+50q+c.$$

2. Given $MR=R'(x)=1500-4x-3x^2$

$$\int R'(x)=\int(1500-4x-3x^2)dx$$

$$R(x) = 1500x - 2x^2 - x^3 + c$$

When $x = 0, R = 0 \Rightarrow c = 0$

$$R(x) = 1500x - 2x^2 - x^3$$

$$Average\ revenue\ function = \frac{R(x)}{x}$$

$$Average\ revenue\ function = \frac{1500x - 2x^2 - x^3}{x}$$

$$Average\ revenue\ function = 1500 - 2x - x^2$$

3. The given marginal revenue (MR)= $10 + 3x + x^2$

$$Total\ revenue\ (TR) = \int (10 + 3x - x^2)dx = 10x + \frac{3}{2}x^2 + \frac{1}{3}x^3 + c$$

At $MR = 0, c = 0$ Then $TR = 10x + \frac{3}{2}x^2 + \frac{1}{3}x^3$

For $x = 2000, TR = 10(200) + \frac{3}{2}(200)^2 + \frac{1}{3}(200)^3$

$$TR = 20,000 + 60,000 + 2,666,666.67 = 2,746,666.67\ FRW$$

## Lesson 10. Elasticity of demand and supply

### a) Learning objective:

Use integration to calculate elasticity of demand and supply.

### b) Teaching resources:

Digital materials including calculator, Manila paper, Markers, Calculators, student's book and other reference textbooks to facilitate research, Mathematical models and Internet connection where applicable

### c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they are enough skilled in previous covered lesson of this unit. In addition, they will perform well if they have enough skills on differentiation, indefinite integrals and definite integrals.

### d) Learning activities

- Facilitate students to be organized in small group discussions
- Invite students to work in small group discussions and perform the learning activity 3.4.2 from the student' book
- Ask each group to make summary and prepare a presentation of findings with other groups.
- Invite group members to present their answers to the whole class;
- As a teacher, harmonize the results from presentation. When harmonizing, insist on showing the students that integration is very needed, especially when it comes to economics, accounting and finance.
- Ask students different probing questions that will lead them to discover the use of integration in calculation the elasticity of demand function
- After attempting different examples of their views, help them to solve given examples in student' s book and be aware of content summary found in student's book
- Guide students to find more economics examples that can be solved with integrals
- After this, invite students to do application activity 3.4.2, assessing their competences and evaluating whether the lesson objectives have been met.

**Answers for learning activity 3.4.2**

a) The first table shows decreasing price associated with increasing quantity, so that is demand function. The second table shows increasing price associated with increasing quantity, so that is the supply function.

b) Price Elasticity of demand is a measure of how responsive a commodity's quantity demanded is to price changes.

c) For both function $q=40$ is associated with $p=40$, the equilibrium price is 40 and equilibrium quantity is 400 units. Notice that we were lucky here, because the equilibrium point is actually one of the points show.

d) The consumer surplus is

$$\int_{0}^{400} (demand)dq - 40(400) = (100)(70+61+46) - (40)(400) = \$7000$$

So the consumer surplus is about $7000

The producer surplus is given

$$(40)(400) - \int_0^{400} (\text{supply})\,\text{dq} = (40)(400) - (100)(14 + 21 + 28 + 33) = \$6,400$$

## e) Answers for Application activity 3.4.2

**Given elasticity of demand** $= \dfrac{4-x}{x}$

We know that $\eta_d = \dfrac{-p}{x}\dfrac{dx}{dp}, \dfrac{-p}{x}\dfrac{dx}{dp} = \dfrac{4-x}{x}$

$$\frac{dx}{4-x} = \frac{-dp}{p} \Rightarrow \int \frac{dx}{4-4x} = -\int \frac{dp}{p}$$

$-\log|4-x| = -\log p + \log c$ Where $c$ is a constant

$$-\log|4-x| = \log\left(\frac{c}{p}\right)$$

$$\log(4-x)^{-1} = \log\left(\frac{c}{p}\right) \Rightarrow \frac{1}{4-x} = \frac{c}{p}, \quad p = c(4-x)$$

When $c = 2 \text{ and } p = 4$

$$4 = c(4-2) \Rightarrow 4 = 2c, \textit{then}, c = 2$$

Therefore, $p = 2(4-x), \textit{also } R = px$ then, $R = 8x - 2x^2$

Hence, the demand function is given by $8-2x$ and revenue function is $8x - 2x^2$

## Lesson 11. Present, Future Values of an Income Stream and Growth Rates

### a) Learning objective:

Use integration to calculate Present, Future Values of an Income Stream and Growth Rates

### b) Teaching resources:

Digital materials including calculator, Manila paper, Markers, Calculators, student's book and other reference textbooks to facilitate research, Mathematical models and Internet connection where applicable.

### c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they are enough skilled in previous covered lesson of this unit. In addition, they will perform well if they have enough skills on differentiation, indefinite integrals and definite integrals.

### d) Learning activities

- Facilitate students to be organized in small group discussions
- Invite students to work in small group discussions and perform the learning activity 3.4.3 from the student' s book
- Ask each group to make summary and prepare a presentation of findings to other groups.
- Invite group members to present their findings to the whole class;
- As a teacher, harmonize the results from presentation. When harmonizing, insist on showing the students that integration is very needed, especially when it comes to economics, accounting and finance.
- Ask students different probing questions that will lead them to discover the use of integration in calculation present, future values of an income stream and Growth Rates
- After attempting different examples of their views, help them to solve given examples in student' s book and be aware of content summary found in student's book
- Guide students to find more examples related to present, future values of an income stream and Growth Rates that can be solved with integrals
- After this, invite students to do application activity 3.4.3, assessing their competences and evaluating whether the lesson objectives have been met.

### Answers for learning activity 3.4.3

If we assume that the income will come in continuously over the 8 years. We assume also that interest will remain constant that period of 8 years.

Let $t$ be the time in years since the purchase of machine. The income from the machine is different depending on the time. From $t=0 \text{ to } t=1(\text{first year})$ the income is \$7000 per year. From $t=1 \text{ to } t=8$ , the income is increasing by \$800 each year. The income follow the function $F(t)=7000+800(t-1)=6200+800t$

To find present value $(PV)$ we need to divide integral into two pieces of functions

$$PV=\int_0^1 7000e^{-0.017t}dt+\int_1^8 (6200+800t)e^{-0.017t}dt=70166$$

The Present value is greater than the cost of the machine. So, the company should by the machine

### e) Answers for application activity 3.4.3

Given $P=10,000,\ r=\frac{5}{100}=0.05\ and\ N=5$

Amount after 5 years is given by $\int_0^5 10,000e^{0.05t}dt$ therefore, $A=\int_0^N Pe^{rt}dt$

$$=10,000\int_0^5 e^{0.05t}dt=10,000\left(\frac{e^{0.05t}}{0.05}\right)_0^5=\frac{10,000}{0.05}\left(e^{0.05t}\right)_0^5$$

$$=200,000\left(e^{0.05(5)}-e^{0.05(0)}\right)=200,000\left(e^{0.25}-1\right)$$

$$=200,000(1.284-1)=200,000(0.284)=56,800$$

Hence the amount after 5 years will be 56,800FRW

## 3.6. Summary of the unit 3

### 1. Anti-derivative

For a continuous function of variable $x$, $y=f(x)$ is anti-derivative of $f(x)$ any function $F(x)$ such that $F'(x)=f(x)$. For any arbitrary $c$, $F'(x)+\text{c}$ is also an anti-derivative of $f(x)$ because $\left(F'(x)+\text{c}\right)'=F'(x)$

1. Indefinite integral

Let $y=f(x)$ be a continuous function of variable $x$. The indefinite integral of $f(x)$ is the set of all its ant-derivatives. If $F(x)$ is any anti-derivative of function $f(x)$, then the indefinite integral of $f(x)$ is denoted as $\int f(x)dx=F(x)+c$

, where c is an arbitrary constant called constant of integration. Thus, $\int f(x)dx = F(x)+c$ if and only if $F'(x) = f(x)$

**2. Properties of indefinite integral**

i) The integral of the product of a constant by a function is equal to the product of the constant by the integral of the function, $\int kf(x) = k\int f(x)dx$

ii) The integral of a sum or difference of functions equal the sum or difference of the integrals of the function, $\int [f(x) \pm g(x)]dx = \int f(x)dx + \int g(x)dx$

iii) The derivative of the indefinite integral is equal to the function to be integrated, $\frac{d}{dx}\int f(x)dx = f(x)$

Basic integration formulae

1. If $k$ is constant, $\int kdx = kx + c$
2. $\int u^n du = \frac{u^{n+1}}{n+1} + c, \quad n \neq -1$
3. If $\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + c, \; for\ x\ nonzero$
4. $\int e^x dx = e^x + c$, the integral of exponential function of base $e$
5. If $a > 0$ and $a \neq 1$, $\int a^x dx = \frac{a^x}{\ln a} + c$

**3. Techniques of integration of indefinite integrals**

**a) Integration by substitution**

It is the method in which the original variables are expressed as functions of other variables. If $F(x) = \int f(x)dx$, the indefinite integral can be obtained by resorting to transformation. If you take $x = g(y)$, then $\int f(x)dx = \int [g(y)]g'(y)dy$. See that $x = g(y) \Rightarrow dx = g'(y)dy$.

**b) Integration by parts**

Let us return to the differentiation of product of two functions $u = f(x)$ and $v = f(x)$

$d(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$. From this, we can integrate both sides with respect to $x$ to get

$$uv = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx \Rightarrow uv = \int vdu + \int udv$$

Therefore,

$\int udv = uv - \int vdu$ or $\int vdu = uv - \int udv$ This is the formula for integration by parts.

**Definite integral**

The definite integral of the function $f(x)$ over the interval $[a,b]$ is expressed symbolically as $\int_a^b f(x)dx$, read as the integral of $f$ with respect to $x$ from $a$ to $b$. The smaller number $a$ is termed the lower limit and $b$, the upper limit of the integration.

## 4. Fundamental theorem of calculus

If $f(x)$ is continuous function on a closed interval $[a,b]$ then fundamental theorem of calculus displayed as $\int_a^b f(x)dx = F(b) - F(a)$. Therefore, if $f(x)$ and $g(x)$ are continuous functions on a closed interval $[a,b]$

## 5. Properties of definite integral

If $f(x)$ and $g(x)$ are continuous functions on a closed interval $[a,b]$ then,

1. $\int_a^b 0dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ (Permutation of bounds)

3. $\int_a^b [\alpha f(x) \pm \beta g(x)] dx = \alpha \int_a^b f(x)dx \pm \int_a^b g(x)dx, \alpha \text{ and } \beta \in \mathbb{R}$ (linearity)

4. $\int_a^a f(x)dx = 0$ (bounds are equal)

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b$ (Charles relation)

6. $\forall x \in [a,b], f(x) \le g(x) \Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx$. It follows that

$f(x) \ge 0 \Rightarrow \int_a^b f(x)dx \ge 0$ and

$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$ positivity

$$\int_{-a}^{b} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{if } f(x) \text{ is even function} \\ 0, \text{if } f(x) \text{ is odd function} \end{cases}$$

### 6. Techniques of integration of definite integrals

#### a) Integration by substitution

The method in which we change the variable to some other variable is called "**Integration by substitution**".

When definite integral is to be found by substitution then change the lower and upper limits of integration. If substitution is $\varphi(x)$ and lower limit of integration is *a* and upper limit is *b* then new lower and upper limits will be $\varphi$(a) and $\varphi$(b) respectively.

#### b) Integration by parts

To compute the definite integral of the form $\int_a^b f(x)g(x)dx$ using integration by parts, simply set $u = f(x)\, and\, dv = g(x)dx$. Then $du = f'(x)dx$ and $v = G(x)$, anti-derivative of $g(x)$ so that the integration by parts becomes $\int_a^b udv = [uv]_a^b - \int_a^b vdu$

**c) Decomposition or simple fractions**

The partial fraction decomposition method is useful for integrating proper rational functions. Divide our more complex rational fraction into smaller, and more easily integrated rational functions. When splitting up the rational function, the rules to follow are the same as from the indefinite integral. Here are the steps for evaluating definite integrals using the method of partial fractions

**step 1:** Factor the denominator of the integrand

**step2:** Split the rational function into a sum of partial fractions

**step3:** Set partial fractions decomposition equal to the original function

**step 4:** solve for numerators of each of the partial fractions

**step 5:** Take the definite integral of each of the partial fractions, and sum together

**7. Application of integration**

**a) Calculation of marginal cost, revenues and profits**

If $C$ denotes the total cost and $M(x) = \frac{dC}{dx}$ is the marginal cost, the cost function given by $C = C(x) = \int M(x)dx + k$, where the constant of integration $k$ represents the fixed cost. On the other hand, Profit function is given by $P(x) = R(x) - C(x)$

Elasticity of demand and supply

Elasticity of the function $y = f\ (x)$ at a point $x$ is defined as the limiting case of ratio of the relative change in $y$ to the relative change in $x$. $\eta = \frac{E_y}{E_x} = \lim_{\Delta x \to 0} \frac{dy}{y} \times \frac{x}{dx} = \frac{xdy}{ydx}$

therefore, $\eta = \frac{xdy}{ydx}$ Elasticity of demand $\eta_d = \frac{-pdx}{xdp}$,

$\frac{-dp}{p} = \frac{dx}{x}.\frac{1}{\eta_d}$ integrating both side with respect $x$ yields $-\int \frac{dp}{p} = \frac{1}{\eta_d}\int \frac{dx}{x}$.

The equation yields the demand function '$p$' as a function of $x$. The revenue function can be found out by using integration.

**b) Present, Future Values of an Income Stream and Growth Rates**

Suppose money can earn interest at annual interest rate of $r$, compounded continuously. Let $F(t)$ be a continuous income function (in dollars per year) that applies between year $0$ *and year* $T$. Then the present value of that income stream is given by $PV = \int_0^T F(t)e^{-rt}dt$

## 3.7 Additional Information for Teacher

For the educative action of the teacher to be effective (in order to respond to all aspects of the students' needs), it is worth mentioning that the teacher needs a wide range of skills, attitudes, a rich and deep understanding of the mathematics subject matter and the pedagogical processes to develop the understanding that is required from the student. It is therefore, imperative for the teacher to not limit himself/herself to the only to the present book, but also to consider getting information from other relevant books, such as those mentioned in the reference. Here the teacher has to emphasize the application of integral in solving problems related to economics, accounting, production and in real life situations. In addition, teachers should be aware of economics terms for better found more applications of integration in economics.

### 3.8 End unit assessment

This part provides the answers of end unit assessment activities designed in integrative approach to assess the key unit competence with cross reference to the textbook.

1.a) $\int(10t^{-3}+12t^{-9}+4t^{3})dt=-\frac{10}{2}t^{-2}-\frac{12}{8}t^{-8}+t^{4}+c$

$=-5t^{-2}-\frac{3}{2}t^{-8}+t^{4}+c$

b) $\int\frac{x^{8}-6x^{5}+4x^{3}-2}{x^{4}}dx=\int(\frac{x^{8}}{x^{4}}-\frac{6x^{5}}{x^{4}}+\frac{4x^{3}}{x^{4}}-\frac{2}{x^{4}})dx=\int(x^{4}-6x+\frac{4}{x}-2x^{-4})dx$

$=\frac{1}{5}x^{5}-3x^{2}+4\ln|x|+\frac{2}{3}x^{-3}+c$

c) $\int_{0}^{1}\frac{xe^{x}}{(x+1)^{2}}dx=\int_{0}^{1}\frac{(x+1)e^{x}}{(x+1)^{2}}dx=\int_{0}^{1}\frac{e^{x}}{(x+1)}dx-\int_{0}^{1}\frac{e^{x}}{(x+1)^{2}}dx$

Integration by part we gat,

$=\left[\frac{e^{x}}{x+1}\right]_{0}^{1}-\int_{0}^{1}(-1)(x+1)^{-2}e^{x}dx-\int_{0}^{1}\frac{e^{x}}{(x+1)^{2}}dx$

$=\left[\frac{e}{2}-1\right]+\int_{0}^{1}\frac{e^{x}}{(x+1)^{2}}dx-\int_{0}^{1}\frac{e^{x}}{(x+1)^{2}}dx$

Therefore, $\int_{0}^{1}\frac{xe^{x}}{(x+1)^{2}}dx=\frac{e}{2}-1$

d) $\int xe^{x}dx$, using the integration by parts

let $u=x\rightarrow du=dx, and\ dv=e^{x}dx\rightarrow v=e^{x}$

$\int xe^{x}dx=uv-\int vdv=xe^{x}-\int e^{x}dx$

$=xe^{x}-e^{x}+c$

$=e^{x}(x-1)+c$

e) $\int_{1}^{e}x^{3}\log xdx$, let $u=\log x\rightarrow du=\frac{1}{x}dx, and\ dv=x^{3}dx\rightarrow v=\frac{x^{4}}{4}$

$$\int_1^e x^3 \log x dx = \int u dv$$

$$= uv - \int v du$$

$$= \frac{x^4}{4}\log x - \frac{1}{4}\int x^3 dx$$

$$= \frac{x^4}{4}\log x - \frac{1}{4}\left(\frac{x^4}{4}\right) + c$$

$$= \frac{x^4}{4}\left(\log x - \frac{1}{4}\right) + c$$

2) Answers varies according to respondents

3) $MC = \frac{x}{\sqrt{x^2+1600}}$ let $t = \sqrt{x^2+1600} \to t^2 = x^2 + 1600$ thus, $xdx = tdt$

$$C(x) = \int \frac{x}{\sqrt{x^2+1600}} dx = \sqrt{x^2+1600} + c$$

Given that the Fixed Cost is 500FRW we get:

$$500 = 400 + c \to c = 100$$

Total cost function $C(x) = \sqrt{x^2+1600} + 100$

An average cost $AC = \frac{C(x)}{x} = \frac{\sqrt{x^2+1600}+100}{x} = \sqrt{1 + \frac{1600}{x^2}} + \frac{100}{x}$

4.

Given $MR = 20 - 5x + 3x^2$

$$\frac{dR}{dx} = 20 - 5x + 3x^2$$

$$\int dR = (20 - 5x + 3x^2)dx$$

$$R = 20x - \frac{5x^2}{2} + \frac{3x^3}{3} + k$$

When $x = 0, R = 0 \Rightarrow K = 0$

Therefore, $R = 20x - \frac{5x^2}{2} + x^3$

5. $I(t) = 6\sqrt{t}$ , using the integration formula $K = \int_a^b I(t)dt$

We have $K = \int_4^9 6\sqrt{t}dt = 6\int_4^9 t^{\frac{1}{2}}dt = \left.\frac{12t^{\frac{3}{2}}}{3}\right|_4^9 = 4\left.\left(\sqrt{t}\right)^3\right|_4^9 = 4(27-8) = 76$

6. The given $p = 1800 - 0.6q^2$ and $MR = 1,800 - 1.8q^2$

i) $TR$ when $q$ is 10 will be $\int_0^{10} MRdq = \int_0^{10}\left(1,800 - 1.8q^2\right)dq$

$= \left[1,800q - 0.6q^3\right]_0^{10} = 18,000 - 600 = 17,400$

ii) The change in $TR$ when $q$ increases from 10 to 20 will be

$$\int_{10}^{20} MRdq = \int_{10}^{20}\left(1,800 - 1.8q^2\right)dq = (36,000 - 4,800) - (18,000 - 600) = 13,800$$

iii) Consumer surplus when $q$ is 10 will be the definite integral of the demand function minus total revenue actually spent by consumers. It will be

$\int_0^{10}(1,800 - 0.6q^2) = \left[1,800q - 0.2q^3\right]_0^{10} = 18,000 - 200 = 17,800$ and

$TR = pq = 1,800q - 0.6q^3 = 18,000 - 600 = 17,400$

Therefore consumer surplus is $17,800 - 17,400 = 400$

## 3.9 Additional activities

1. The total cost C(x) in Rupees, associated with the production of x units of an item is given by: $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$, find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level output.

**solution:**

Since marginal cost is the rate of change of total cost with respect to the output, we have: marginal cost: $MC = \frac{dC}{dx} = 0.005\left(3x^2\right) - 0.02\left(2x\right) + 30$;

when $x = 3$, $MC = 0.015\left(3^2\right) - 0.04\left(3\right) + 30 = 30.015$

2. Calculate:

$\int xe^x dx$

**Answer:**

$I = \int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c = e^x\left(x-1\right) + c$

### 3.9.1. Remedial activities

Assume the rate of investment is given by the function $I(t) = \ln t$ compute the total capital accumulation between the 1st and the 5th years.

**Solution**

To calculate the capital accumulation, we use the formula

$K = \int_a^b I(t)dt = \int_1^5 \ln t dt$, integrating by parts, let $u = \ln t, \rightarrow du = \frac{dt}{t}, and\ dv = dt \rightarrow v = t$

we have $\int \ln t dt = t\ln t - \int t\frac{dt}{t} = t\ln t - \int dt = t\ln t - t$

hence , $K = (t\ln t - t)\Big|_1^5 = (5\ln 5 - 5) - (\ln 1 - 1) = 5\ln 5 - 4 = 4.05$

### 3.9.2 Consolidation activities

Evaluate the following integrals:

a) $\int \frac{xdx}{\sqrt{1+x}}$

b) $I = \int \frac{e^x + 1}{e^x - 1} dx$

**Solution**

$$\int \frac{(1+x)-1}{\sqrt{1+x}}dx \Rightarrow \int \frac{1+x}{\sqrt{1+x}}dx - \int \frac{dx}{\sqrt{1+x}}$$

$$\Rightarrow \int (1+x)^{\frac{1}{2}}dx - \int (1+x)^{-\frac{1}{2}}dx \Rightarrow \frac{3}{2}(1+x)^{\frac{3}{2}} - 2(1+x)^{\frac{1}{2}} + c$$

$b.\ I = \int \frac{e^x+1}{e^x-1}dx$

**Solution**

$$\frac{e^x+1}{e^x-1} = 1 + \frac{2}{e^x-1} = 1 + 2\left(\frac{1}{e^x\left(1-e^{-x}\right)}\right) = 1 + 2\left(\frac{e^{-x}}{1-e^{-x}}\right)$$

$$I = \int dx + 2\int \frac{e^{-x}}{1-e^{-x}}dx$$

$$I = x + 2\ln\left|1-e^{-x}\right| + c$$

### 3.9.3 Extended activities

$$\int_0^3 6xe^{x^2+1}dx$$

**Solution**

Let $x^2+1=t,\ 2xdx=dt \rightarrow xdx=\frac{dt}{2}$

When $x=0, t=1\ and\ when \in=3, t=10$

$$\int_0^3 6xe^{x^2+1}dx = \int_1^{10} 6e^t\frac{dt}{2} = 3\int_1^{10} e^t dt = 3\left[e^t\right]_1^{10} = 3\left(e^{10}-e\right)$$

# UNIT 4 INDEX NUMBERS AND APPLICATIONS

## 4.1 Key unit competence

Apply index numbers in solving financial related problems, interpreting a value index, and drawing appropriate decisions.

## 4.2 Prerequisite

The students will perform well in this unit if they have a good background on univariate and bivariate statistics learnt in senior five.

## 4.3 Cross-cutting issues to be addressed

- Inclusive education (promoting education for all in teaching and learning process)
- Peace and value Education (respect the views and thoughts of others during class discussions)
- Gender: Provide equal opportunity for boys and girls to participate in class

## 4.4 Guidance on introductory activity 4

- Invite students to work in small group discussions, discuss and perform the introductory activity 4 found in student's book;
- Facilitate students' discussions and ask them to avoid noise or other unnecessary conversations;
- During discussions, let students think of different ways to solve the given problem;
- Walk around in all groups to provide assistance where needed;
- Invite group members to present their findings and encourage both boys and girls to actively participate in presentations;
- Let other students share complement on the presentations of their classmates

- Basing on students' experience, prior knowledge and abilities shown in answering the questions for this introductory activity, use different probing questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

## Answer for introductory activity 4

i) This standard of living is not said to have risen 12 times during this period because index of living change over time. He should use $P = \frac{\sum P_t}{\sum P_0} \times 100$ to find the standard living in the current year.

ii) We use index numbers to detect changes in a variable (rising or decreasing)

iii) We can take the income over the cost of living index

iv) The method that can be used to detect changes in prices are discussed below

| Price indices | |
|---|---|
| Un-weighted index | Weighted index |
| • Simple aggregative method<br>• Simple average of price relatives method | • Weighted aggregative method<br>• Weighted average of price relatives method |

**Un-weighted index: w**eights are not assigned to the various items used in the calculation of the un-weighted index number. These include:

- **Simple Aggregate Method:** This method assumes that different items and their prices are quoted in the same units and is given by $P = \frac{\sum P_t}{\sum P_0} \times 100$

- **Simple Average of Price Relatives Method:** This method is superior to the previous one because it is unaffected by the unit in which the prices of various commodities are quoted. Because the price relatives are pure numbers, they are independent of the original units in which they are quoted. The price index number is defined using price relatives as follows:

$$P = \frac{\sum \frac{P_t}{P_0} \times 100}{N}$$

**Weighted Index Number:** All items or commodities are given rational weights in a weighted index number. These weights indicate the relative importance of the items used in the index calculation. In most cases, the quantity of usage is the most accurate indicator of importance.

- **Weighted Aggregative Price Indices:** Weights are assigned to each item in the basket in various ways in weighted aggregative price indices, and the weighted aggregates are also used in various ways to calculate an index. In most cases, the price index number is calculated using the quantity of usage. It is given by $P=\frac{\sum P_t q_0}{\sum P_0 q_0}\times 100$

- **Weighted Price Relative Method:** Under this method price index is constructed on the basis of price relatives and not on the basis of absolute prices. The price index is obtained by taking the average of all weighted price relatives. It is given by

$$P(weighted\ arithmetic\ mean)=\frac{\sum W\left(\frac{P_1}{P_0}\times 100\right)}{\sum W}$$

## 4.5 List of lessons

| Headings | # | Lesson title/ sub-headings | Learning objectives | Number of periods |
|---|---|---|---|---|
| 4.1 Introduction to index numbers | Introductory activity | | Arouse the curiosity of students on the content of unit 3. | 1 |
| | 1 | Meaning, types and characteristics of index number | Differentiate between types and characteristics of index numbers | 3 |
| | 2 | Construction of indices | Use index number to interpret a Laspeyer's and Paasche's price index | 3 |
| **4.2 End unit assessment** | | | | **2** |

### Lesson 1: Meaning, types and characteristics of index number

#### a) Learning objective:

Differentiate between types and characteristics of index numbers

#### b) Teaching resources:

Student's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with internet connection and mathematical software like Microsoft Excel

#### c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they are skilled enough in the content of statistics learnt in senior five.

**d) Learning activities**

- Facilitate students to be organized in small group discussions
- Invite Students to work in their respective small group discussions and perform the learning activity 4.1.1 found in senior six Mathematics student's book;
- Move around for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class and let other students give their comments on different presentations.
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills on index numbers.
- Ask students to do the application activity 4.1.1 found in student's book and evaluate whether lesson objectives were achieved to assess their competences.

**Answers for learning activity 4.1.1**

a) Statistical measure that can be used by this businessman to detect changes in a variable or group of variables in his business is index number. Index number is a statistical measure used to detect changes in a variable or group of variables.

b) He should refer to the following formula to calculate consumer price index:

Consumer price index is given by: $cost\ of\ living\ index = \frac{\sum WP}{\sum W}$ where, $P = \frac{P_1}{P_0} \times 100$ and $W$ are weights

**e) Answers for the application activity 4.1.1**

1. Index numbers measures the pulse of the economy and act as a barometer to find the variations economic condition of the country. Hence index number acts as a economic barometer.

2. Characteristics of index numbers

Index numbers have the following important characteristics:

- Index numbers are of the type of average that measures the relative changes in the level of a particular phenomenon over time. It is a special type of average that can be used to compare two or more series made up of different types of items or expressed in different units.

- Index numbers are expressed as percentages to show the level of relative change.
- Index numbers are used to calculate relative changes. They assess the relative change in the value of a variable or a group of related variables over time or between locations.
- Index numbers can also be used to quantify changes that are not directly measurable. For example, the cost of living, price level, or business activity in a country are not directly measurable, but relative changes in these activities can be studied by measuring changes in the values of variables/factors that affect these activities.

## Lesson 2: Construction of indices

**a) Learning objectives:**

Use index number to interpret a Laspeyer's and Paasche's price index

**b) Teaching resources:**

Student's book and other Reference books to facilitate research, scientific calculator, Manila paper, markers, pens, pencils, if possible computers with internet connection and mathematical software Microsoft Excel.

**c) Prerequisites/Revision/Introduction:**

Students will perform well in this lesson if they are skilled enough in statistics learnt in senior five and content of previous lesson

**d) Learning activities**

- Invite students to work in pairs and perform the learning activity 4.1.2 found in senior six mathematics student' book.
- Move around in the class for facilitating students where necessary.
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a teacher, harmonize findings from students' presentation;
- Use different probing questions and guide students to explore the content and examples given in the students' book and lead them to construct index number
- Invite students to apply steps of constructing index numbers
- After this step, guide students to do the application activity 4.1.2 found

in student's book and evaluate whether lesson objectives were achieved.

**Answers of learning activity 4.1.2**

a) The prices for cheese(100g) increased in 1998, 1999 and 2000 while prices for Egg and potatoes increased in 1999 and decreased in 2000

b) The methods that can be used to detect changes in these variable are divided into three broad categories, as shown below:

| Price indices | |
|---|---|
| Un-weighted index | Weighted index |
| • Simple aggregative method<br>• Simple average of price relatives method | • Weighted aggregative method<br>• Weighted average of price relatives method |

**Un-weighted index: w**eights are not assigned to the various items used in the calculation of the un-weighted index number. These include:

- **Simple Aggregate Method:** This method assumes that different items and their prices are quoted in the same units and is given by $P = \frac{\sum P_t}{\sum P_0} \times 100$
- **Simple Average of Price Relatives Method:** This method is superior to the previous one because it is unaffected by the unit in which the prices of various commodities are quoted. Because the price relatives are pure numbers, they are independent of the original units in which they are quoted. The price index number is defined using price relatives as follows:

$$P = \frac{\sum \frac{P_t}{P_0} \times 100}{N}$$

- **Weighted Index Number:** All items or commodities are given rational weights in a weighted index number. These weights indicate the relative importance of the items used in the index calculation. In most cases, the quantity of usage is the most accurate indicator of importance.
- **Weighted Aggregative Price Indices:** Weights are assigned to each item in the basket in various ways in weighted aggregative price indices, and the weighted aggregates are also used in various ways to calculate an index. In most cases, the price index number is calculated using the

quantity of usage. It is given by $P=\frac{\sum P_t q_0}{\sum P_0 q_0}\times 100$

- **Weighted Price Relative Method:** Under this method price index is constructed on the basis of price relatives and not on the basis of absolute prices. The price index is obtained by taking the average of all weighted price relatives. It is given by

$$P(weighted\ arithmetic\ mean)=\frac{\sum W\left(\frac{P_1}{P_0}\times 100\right)}{\sum W}$$

a) The simple aggregative index for the year 1999 over the year 1998 given by

$$\frac{\sum P_n}{\sum P_0}=\frac{24.60}{20}\times 100=123$$

b) The Simple aggregative for the year 2000 over the year 1998

given by $\frac{\sum P_n}{\sum P_0}=\frac{24.60}{20}\times 100=123$

## e) Answers of application activity 4.1.2

$\exp enditure = price \times quantity$

| items | $P_0$ | $P_0Q_0$ | $P_1$ | $P_1Q_1$ | $Q_0$ | $Q_1$ | $P_0Q_1$ | $P_1Q_0$ |
|---|---|---|---|---|---|---|---|---|
| A | 50 | 100 | 60 | 180 | 2 | 3 | 150 | 120 |
| B | 40 | 120 | 40 | 200 | 3 | 5 | 200 | 120 |
| C | 100 | 100 | 120 | 12 | 1 | 1 | 100 | 120 |
| D | 20 | 80 | 25 | 100 | 4 | 4 | 80 | 100 |
| Total | | 400 | | 600 | | | 530 | 460 |

Laspeyers price index is $P01 = \frac{\sum P_1Q_0}{\sum P_0Q_0} \times 100 = \frac{460}{400} \times 100 = 115$

Paasche's price index number is $P01 = \frac{\sum P_1Q_1}{\sum P_0Q_1} \times 100 = \frac{600}{530} \times 100 = 113.21$

## 4.6. Summary of the unit 4

### 1. Meaning of index number

An index number is a statistical measure used to detect changes in a variable or group of variables. In addition, a single ratio (or percentage) that measures the combined change of several variables between two different times, places, or situations is referred to as an index number. Index numbers are expressed as percentages.

### 2. Characteristics of index numbers

Index numbers have the following important characteristics:

- Index numbers are of the type of average that measures the relative changes in the level of a particular phenomenon over time. It is a special type of average that can be used to compare two or more series made up of different types of items or expressed in different units.
- Index numbers are expressed as percentages to show the level of relative change.
- Index numbers are used to calculate relative changes. They assess the relative change in the value of a variable or a group of related variables over time or between locations.
- Index numbers can also be used to quantify changes that are not directly measurable. For example, the cost of living, price level, or business activity in a country are not directly measurable, but relative changes in these activities can be studied by measuring changes in the values of variables/factors that affect these activities.

### 3. Types of index numbers

#### a) Consumer price index

A consumer price index (CPI) tracks price changes in a basket of consumer goods and services purchased by households. CPI measures changes in the price level for a specific group of consumers in a given region. CPI can be calculated for industrial workers, city workers and agricultural workers. Consumer price index is given by:

$cost\ of\ living\ index = \frac{\sum WP}{\sum W}$ where, $P = \frac{P_1}{P_0} \times 100$ and $W$ are weights

**b) The Producer Price Index**

The Producer Price Index (PPI) tracks the average change in selling prices received by domestic producers for their output over time. For many products and services, the prices included in the PPI are from the first commercial transaction. The Producer Price Index chart alerts the market to changes in the prices of products leaving the producers. PPIs are available for several manufacturing and service industries' output.

**c) Wholesale Price Index number**

The Wholesale Price Index (WPI) is the wholesale price of a representative basket of goods. The wholesale price index number reflects the general price level change. It lacks a reference consumer category, unlike the CPI. For example, the WPI with 2011 as a baseline is 156 in March 2014, indicating that the general price level has risen by 56% during this time.

**d) Industrial production index**

The industrial production index measures the change in the level of industrial production over a given time period across multiple industries. It represents a weighted average of quantity relatives. The index of industrial production provided by

$$P = \frac{\sum q_1 w}{\sum w}$$

## 4.7 Additional Information for Teacher

For the teacher to be effective (in order to respond to all aspects of the students' needs), it is worth mentioning that the teacher needs a wide range of skills, attitudes and values, a rich and deep understanding of the mathematics subject matter and the pedagogical processes to develop the understanding that is required from the student. It is therefore, imperative for the teacher to not limit himself/herself to the only to the present book, but also to consider getting information from other relevant books, such as those mentioned in the reference. Here the teacher has to emphasize the application of index numbers in solving problems related to economics, accounting, production and in real life situations. In addition, teachers should be aware of economics terms for better found more applications of matrices and determinants.

## 4.8 End unit assessment

1.

**Solution**

| Items | Weight | $P_0$ | $P_t$ | $P=\left(\frac{P_t}{P_0}\right)\times 100$ | $PW$ |
|---|---|---|---|---|---|
| Food | 10 | 150 | 225 | 150 | 1500 |
| House rent | 5 | 50 | 150 | 300 | 1500 |
| Clothing | 2 | 30 | 60 | 200 | 400 |
| Fuel | 3 | 30 | 75 | 250 | 750 |
| Others | 5 | 50 | 75 | 150 | 750 |
| Total | 25 | | | | 4900 |

$$CPI=\frac{\sum WP}{\sum W}=\frac{4900}{25}=196$$

2.

| Commodity | Base year | | Current year | | $p_0q_0$ | $p_1q_0$ | $p_0q_1$ | $p_1q_1$ |
|---|---|---|---|---|---|---|---|---|
| | $P_0$ | $q_0$ | $P_1$ | $q_1$ | | | | |
| A | 10 | 5 | 20 | 2 | 50 | 100 | 20 | 40 |
| B | 15 | 4 | 25 | 8 | 60 | 100 | 120 | 200 |
| C | 40 | 2 | 60 | 6 | 80 | 120 | 240 | 360 |
| D | 25 | 3 | 40 | 4 | 75 | 120 | 100 | 160 |
| Total | | | | | 265 | 440 | 480 | 760 |

i) Laspeyre's given by $\frac{\sum p_1q_0}{\sum p_0q_0}\times 100=\frac{440}{265}\times 100=166.04$

ii) Paasche's given by $\frac{\sum p_1q_1}{\sum p_0q_1}\times 100=\frac{700}{480}\times 100=158.3$

3. Information will differ from the results collected from local market. Note: Teacher make harmonization referring to the formula

## 4.9 Additional activities

### 4.9.1 Remedial activities

Compute Laspeyre's and Paasche's Index numbers for 2000 from the following data.

| Commodity | P0 | P1 | Q0 | Q1 | P0Q0 | P1Q0 | P0Q1 | P1Q1 |
|---|---|---|---|---|---|---|---|---|
| A | 400 | 85 | 100 | 120 | 40000 | 85000 | 48000 | 102000 |
| B | 320 | 690 | 20 | 60 | 640 | 13800 | 19200 | 41400 |
| C | 720 | 1600 | 10 | 10 | 7200 | 16000 | 7200 | 16000 |
| D | 720 | 2100 | 10 | 20 | 7200 | 21000 | 14400 | 42000 |
| Total | | | | | 60800 | 135800 | 88800 | 201400 |

Laspeyer's quantity index number is $\frac{\sum Q1P0}{\sum Q0P0}\times 100=\frac{88800}{60800}\times 100=146.05$

Paasche's index number is $\frac{\sum Q1P0}{\sum Q0P0}\times 100=\frac{88800}{60800}\times 100=146.05$

### 4.9.2 Consolidation activities

Find the price index number using simple aggregate method in the following prices in rupees , use 1995 as base year in the problem

| Commodity | P | Q | R | S | T |
|---|---|---|---|---|---|
| Price in 1995 | 15 | 20 | 24 | 22 | 28 |
| Price in 2000 | 27 | 38 | 32 | 40 | 45 |

**Solution**

| Commodity | Price in 1995 (base year) | Price in 2000 (current year) |
|---|---|---|
| P | 15 | 27 |
| Q | 20 | 38 |
| R | 24 | 32 |
| S | 22 | 40 |
| T | 28 | 45 |
| Total | 109 | 180 |

From the table $\sum P0 = 109$ , $\sum P1 = 109$

Price index number (P01) $(P01) = \frac{\sum P1}{\sum P0} \times 100 = \frac{182}{109} \times 100 = 166.97$

### 4.9.3. Extended activities

1. Find $x$ if the price index number by simple aggregate method is 125 (The prices are in rupees)

| Commodity | P | Q | R | S | T |
|---|---|---|---|---|---|
| Base year price | 8 | 12 | 16 | 22 | 18 |
| Current year price | 12 | 18 | $x$ | 28 | 22 |

**Solution**

| Commodity | Base year price | Current year price |
|---|---|---|
| | P0 | P1 |
| P | 8 | 12 |
| Q | 12 | 18 |
| R | 16 | X |
| S | 22 | 28 |
| T | 18 | 22 |
| Total | 76 | X+80 |

From the table, $\sum P0 = 76$, $\sum P1 = x + 80$ , given the price index ( $p_{01}$ ) is 125

Since $p_{01}=\frac{\sum p_1}{\sum p_0}\times 100$

$$125=\frac{x+80}{76}\times 100$$

$$125\times 76=(x+80)\times 100\rightarrow 95=x+80$$

After solving the above equation we get $x=15$

1. The Cost of Living Index Number for years 1995 and 1999 are 140 and 200 respectively. A person earns 11,200 FRW per month in the year 1995. What should be his monthly earnings in the year 1999 in order to maintain his standard of living as in the year 1995?

**Solution**

For the year 1995,

The cost living index $CLI=140$ and income is 11,200FRW

Now, real income $real\ income=\frac{income}{CLI}\times 100$

$$=\frac{11200}{140}\times 100=8000$$

For the year 1999, CLI=200

$$real\ income=\frac{income}{CLI}\times 100$$

$$8000=\frac{income}{200}\times 100$$

$$income=\frac{8000\times 200}{100}=16000$$

In order to maintain the same standard of living as in 1995, income in 1999 should be 16,000FRW

# UNIT 5 | INTRODUCTION TO PROBABILITY

## 5. 1 Key Unit competence

Use probability concepts to solve mathematical and production, financial, and economical related problems and draw appropriate decisions

## 5.2 Prerequisite

Students will perform well in this unit if they make a short revision on the descriptive statistics learnt (in **S5** unit **4** and **5**), Tree and Venn diagram (in **S4**) and previous content for S6. But it is recommendable to revise even algebra and other content in order to improve your level of critical thinking.

## 5.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation and making groups)

## 5.4 Guidance on introductory activity

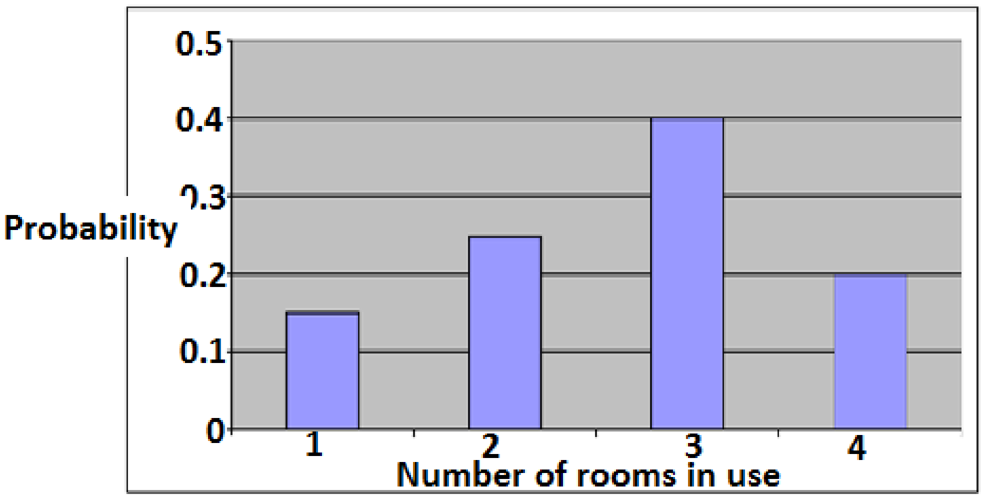
- Form groups of students and invite Students to work on questions for **Introductory activity** found in student's book unit 5;
- Guide students to read and analyse the problem related different cases of the gender that 3children can have: they have to write all those cases on a sheet of paper;
- Guide students to find out many possibilities as they can;
- Invite students with different working steps to present their findings to the whole class discussion;
- Basing on their experience, prior knowledge and abilities shown in answering the questions for this activity , use different questions to guide Students to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

**Answer for introductory activity 5:**

a) Table of probability distribution

| Number of Warehouse in use(x) | Number of days | Probability f(x) |
|---|---|---|
| 1 | 3 | 0.15 |
| 2 | 5 | 0.25 |
| 3 | 8 | 0.40 |
| 4 | 4 | 0.20 |
| | $\sum f = 20$ | $\sum f(x) = 1.00$ |

b) graph of probability distribution




c) $f(x) \geq 0$ , $\sum f(x) = 1$. Therefore, conditions are satisfied

d) Answers are different depending on student. Note: Teacher harmonizes the answers focussing on prediction and decision making.

## 5.5. List of lessons/sub-heading

Use probability concepts to solve mathematical and production, financial, and economical related problems and draw appropriate decisions

| Headings | Number of lessons | Lesson title/ sub-headings | Learning objectives | Number of periods |
|---|---|---|---|---|
| 5.1. Key concepts of probability | Introductory activity 5 | | Arouse the curiosity of students on the content of unit 5 | 1 |
| | 1 | Definitions of probability terminologies | Clarify probability terminologies: probability, likely, unlikely, certain, uncertain, impossible events and sample space and give examples on each key term. | 2 |
| | 2 | Empirical rule, axioms and theorems | Use and apply probability properties to compute the number of possible outcomes of occurrences under equally likely assumptions. | 2 |
| | 3 | Mutual exclusive, mutual exhaustive | Distinguish between mutually exclusive and non-exclusive events. | 1 |

| | | | | |
|---|---|---|---|---|
| | 4 | Independence, and conditional probability | Use different counting techniques to determine the number of possibilities or occurrence outcomes for an event. | 2 |
| | 5 | Successive trials, Tree diagram, Bayes theorem and applications | Apply Bayes theorem to solve problems involving conditional probability in production and finance. | 2 |

| | | | | |
|---|---|---|---|---|
| 5.2. Probability distributions: | 1 | Discrete Probability Distribution | Appreciate the application of probability ( distribution) in production, finance, and economics | 3 |
| | 2 | Binomial Experiment | Appreciate the application of probability ( distribution) in production, finance, and economics | 2 |
| | 3 | Bernoulli distribution | Appreciate the application of probability ( distribution) in production, finance, and economics | 1 |

| | | | | |
|---|---|---|---|---|
| | 4 | Poisson distribution | Appreciate the application of probability (distribution) in production, finance, and economics | 2 |
| | 5 | Continuous Probability Distribution | Appreciate the application of probability (distribution) in production, finance, and economics. | 2 |
| | 6 | Normal distribution | Appreciate the application of probability (distribution) in production, finance, and economics. | 2 |
| 5.3. Applications of probability and probability distributions in production, finance and economics | 1 | Some of applications examples | Appreciate the application of probability in production, finance, and economics. | 5 |
| **5.4. End unit assessment** | | | | **3** |
| **Total period** | | | | **30** |

### Lesson 1: Definitions of probability terminologies.

#### a) Learning objective

Define probability terminologies: probability, likely, unlikely, certain, uncertain, impossible events and sample space and give examples on each key term.

**b) Teaching resources**

Playing cards, graph papers, manila papers, calculators, coin and dice.

**c) Prerequisites/Revision/Introduction:**

Students will perform well in this lesson if they make a short revision on the content learnt as descriptive statistics both univariate and bivariate statistics in S5, unit 4 and 5 respectively and venn diagram from S4.

**d) Learning activities**

- Form small groups of students and instruct them on how to complete the learning activity 5.1.1;
- Walk around to each group and ask pertinent questions about the total number of cards and the number of specified cards;
- Ask each group to share their answers with their neighbouring groups and to encourage each other to improve;
- Invite different working groups to present their findings to the entire class for discussion;
- As a teacher, harmonize their responses by emphasizing that there are several possibilities for selecting a card.
- Use various probing questions to guide them through the various examples provided in the student's book and lead them to explain the main concepts of probability and their definitions of events and their types, outcome, sample space, and so on.
- Following this, guide students through the application activity 5.1.1 and assess whether the lesson objectives were met.

**Answers of learning activity 5.1.1.**

1. **a.** 52
   **b.** 4
   **c.** 1
2. Answers may vary; for example Selecting a red card, Selecting a queen of heart and so on.

**e) Answers of activity 5.1.1.**

1. c)
2. d)

3. $\Omega=\{2,3,4,5,6,7,8,9,10,11,12\}$

## Lesson 2: Empirical rules, axioms and theorems.

**Learning objective:**

Use and apply probability properties to compute the number of possible outcomes of occurrences under equally likely assumptions.

**d) Teaching resources:**

*Manila papers, calculators, coins, dice, etc.*

**e) Prerequisites/Revision/Introduction:**

Students will do well if they have learned the previous lessons for this unit and are proficient in descriptive statistics.

**d) Learning activities**

- Allow students to work in groups to complete the activity 5.1.2;
- Observe each group and ask probing questions to guide students to the correct answer;
- Request students to present their findings in a whole class discussion;
- Request those students present their findings during a whole-class discussion;
- As a teacher, harmonize answers for students and emphasize how to determine the probability of an event using probability properties and formulae.
- Use different probing questions to lead students to explore examples from the student's book and lead them to establish and use probability properties, determine probability of different events: certain event, impossible event, probability of complementary event, mutually exclusive or incompatible event.
- Following this, guide students through the application activity 5.1.2 and assess whether the lesson objectives were met.

**Answer of the learning activity 5.1.2**

1.

a) 11

**b) 4**,

c) $\frac{4}{11}$

**d) 7**,

e) $\frac{7}{11}$

f)

i) Empty set

ii) $\{O, A, I, P, R, B, T, Y\}$ $\{O, A, I, I, P, R, B, B, L, T, Y\}$

iii) $\{P, R, B, B, L, T, Y\}$

iv) $\{O, A, I, I\}$

### e) Answers of application activity 5.1.2

1. a) The probability that the letter M is chosen is $P(M) = \frac{2}{11}$ and

   b) $P(T) = \frac{2}{11}$

2. a) $\frac{13}{19}$ b) $\frac{3}{19}$ c) $\frac{3}{19}$

## Lesson 3: Additional law of probability

a) Learning objective:

Use different counting techniques to determine the number of possibilities or occurrence outcomes for an event.

### b) Teaching resources:

*Manila papers, calculators, coins, dice, etc.*

### c) Prerequisites/Revision/Introduction:

Students will perform well in this lessons if they try to relate the content for all previous lessons for this units to real life problems involving events of chance.

**d) Learning activities**

- Form small groups and give Students instructions about the learning activity. 5.1.3
- Move around to see if every Student is contributing to the provided task.
- Invite group representatives to resent the findings in whole class discussions;
- Teacher harmonizes answers from students of the learning activity. 5.1.3.
- Guide student through a discussion of the addition law of probability, which states that if A and B are mutually exclusive events in a sample space E, then $P(A\cup B)=P(A)+P(B)-P(A\cap B)$. This is known as the probability addition law, from which we can deduce that if A and B are mutually exclusive events, then. $P(A\cup B)=P(A)+P(B)$.
- Lead them to apply the addition law

$$P(E_1\cup E_2\cup E_3\cdots\cup E_n)=P(E_1)+P(E_1)+P(E_3)+\cdots+P(E_n)$$

for mutually exclusive events and understand that the addition law of probability applies to inclusive events

$$P(E_1\ or\ E_2\ or\ E_3\cdots or\ E_n)=P(E_1\cup E_2\cup E_3\cdots\cup E_n)=P(\cup_i^n E_i)$$

$$=\sum_{i=1}^{n}P(E_i)-\sum_{j>i=1}^{n}P(E_i\cap E_j)+\sum_{k>j>i=1}^{n}P(E_i\cap E_j\cap E_k)+\cdots+(-1)^n P(E_1\cap E_2\cap\cdots\cap E_n)$$

- Let them know that, if two events $A$ and $B$ are **exhaustive** $(A\cup B=\Omega)$ then $P(A\cup B)=1$
- Use various probing questions to guide students through the various examples provided in the student book, allowing them to realize that probability is applicable in real life.
- Following this, guide the student through the application activity 5.1.3 and assess whether the lesson objectives were met.

**Answer of learning activity 5.1.3**

The probability that a component selected at random is **either** standard or top quality is $0.65+0.18=0.83$

### e) Application activity 5.1.3

| | |
|---|---|
| 1. $\frac{5}{6}$<br>2. $\frac{1}{2}$ | 3.a) $\frac{3}{8}$<br>b) $\frac{5}{8}$<br>c) $\frac{1}{32}$<br>4. P(intoxicated or accident)<br>=P(intoxicated)+ P(accident)<br>- P(intoxicated and accident)<br>$=0.32+0.09-0.06=0.35$ |

## Lesson 4: Independent, dependent, and conditional probability

### a) Learning objective

Use different counting techniques to determine the number of possibilities or occurrence outcomes for an event.

### b) Teaching resources

Playing cards, graph papers, manila papers, calculators, coin and dice.

### c) Prerequisites/Revision/Introduction:

Students will perform well in this lessons if they try to relate the content for all previous lessons for this units to real life problems involving events of chance and also if there are enough skilled on sets theory covered in S4.

### d) Learning activities

- Form small groups of students, and the teacher will instruct them on how to complete the learning activity 5.1.4.
- Walk around to each group and ask probing questions that lead them to consider the total number of cards and the number of specified cards;
- Ask each group to share their answers with the neighboring group and to support one another for improvement;
- Ask neighbouring groups to share their answers and ask them to support each other where they become more challenged in solving that activity.

- Request students to present their findings in a whole class discussion;
- Invite groups with different working steps to present their findings to the entire class for discussion;
- As a teacher, harmonize their responses by emphasizing that there are multiple ways to choose a card, and highlight how to determine the probability of an event using the classical probability.
- Use different probing questions and guide students to explore **examples** given in the student's book.
- Lead them to know that If probability of event $B$ is not affected by the occurrence of event $A$, events $A$ and $B$ are said to be **independent** and $P(A \cap B) = P(A) \times P(B)$.This rule is the simplest form of the **multiplication law** of probability.
- Lead them know, establish and use of formula for **conditional probability** **of** $B$ **given** $A$ $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$ When the outcome or occurrence of
- the first event affects the outcome or occurrence of the second event (the two events are said to be ***dependent***).
- Emphasize also on the result, that we have general statement of the **multiplication law** write $P(A \cap B) = P(B) \times P(A|B)$.
- After this step, guide students to do the **application activity 5.1.4.** and evaluate whether lesson objectives were achieved.

## Learning activity 5.1.4

1.

   a) Sample space for the first drawing is a set of $52\,cards$ , But for the second drawing the sample space is a set of $51\,cards$ .

   b) The outcomes of the first draw has affected the outcome of the second i.e. the outcome of the second dependent on the first drawing.

## Answer of learning activity 5.1.4

The occurrence of event $B$ is not affected by occurrence of event $A$ because after the first trial the pen is replaced in the box. It means that the sample space does not change.

## e) Application activity 5.1.4

1. $P(\text{red and red}) = \frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$

2. $P(\text{head and } 3) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

3. a) $\frac{1}{7} \times \frac{1}{5} = \frac{1}{35}$ b) $\frac{1}{7} \times \frac{4}{5} + \frac{6}{7} \times \frac{1}{5} = \frac{2}{7}$ c) $\frac{6}{7} \times \frac{4}{5} = \frac{24}{35}$

4. $p(H \text{ and } A) = 0.53 \times 0.27 = 0.1431$

5. $P(6 \mid even) = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$

6. $P(White \mid Black) = \frac{P(Black\ \ and\ \ White)}{P(Black)} = \frac{0.34}{0.47} = 0.72$

## Lesson 5: Successive trials, Tree diagram and Bayes theorem and its applications

### a) Learning objective:

Apply Bayes theorem to solve problems involving conditional probability in production and finance.

### b) Teaching resources:

*Manila papers, calculators, coins, dice, etc.*

### c) Prerequisites/Revision/Introduction:

Students will perform well in this lessons if they try to relate the content for all previous lessons for this units to real life problems involving events of chance and also if there are enough skilled on tree and Venn diagrams.

### d) Learning activities

- Invite students to work in small groups, attempt the learning activity 5.1.5 and answer to related questions;

- Ask neighbouring groups to share their answers and support each other by exchanging arguments on that activity;
- Invite group representatives to resent the findings in a whole class discussions;
- Teacher harmonizes answers for students on the learning **activity 5.1.5.**
- Guide students to discuss and apply a **tree diagram technique** to show the probabilities of certain **outcomes** occurring when two or more **trials** take place in succession.
- Guide Students to discuss and apply **Bayes' formula/ Bayes' rule** to determine probability.

$$P(B_i \mid A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A \mid B_i)P(B_i)}{\sum_{i=1}^{n} P(A \mid B_i)P(B_i)}$$

- Use different probing questions and guide them to explore **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- Let them know that for each **trial,** the number of branches is equal to the number of possible outcomes of that trial.
- Use different probing questions and guide students to explore different **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- After this step, guide students to do the **application activity 5.1.5** and evaluate whether lesson objectives were achieved.

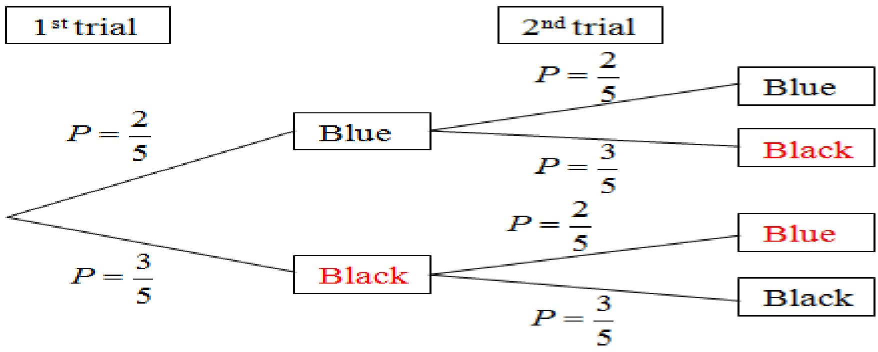
**Answer of learning activity 5.1.5.**

1.

a) Probability of choosing a blue pen is $\frac{4}{10} = \frac{2}{5}$ and probability of choosing a black pen is $\frac{4}{10} = \frac{2}{5}$ .

b) Probabilities on the second trial are equal to the probabilities on the first trial since after the $1^{st}$ trial the pen is replaced in the box.

2. Complete figure ***"Figure to be modified"***




3. $P(A) = P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + P(A/B_3)P(B_3)$

$$P(B_1|A) = \frac{P(B_1 \cap A)}{P(A)} = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

$$P(B_2|A) = \frac{P(B_2 \cap A)}{P(A)} = \frac{P(A|B_2)P(B_2)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

$$P(B_3|A) = \frac{P(B_3 \cap A)}{P(A)} = \frac{P(A|B_3)P(B_3)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

**Generally,**

$$P(B_i|A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{3} P(A|B_i)P(B_i)}$$

## d) Application Activity 5.1.5

1. $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

2. a) $P(3\,\text{boys}) = \frac{10}{16} \times \frac{9}{15} \times \frac{8}{14} = 0.214$

b) $P(2 \text{ boys and } 1 \text{ girl}) = \frac{10}{16} \times \frac{9}{15} \times \frac{6}{14} + \frac{10}{16} \times \frac{6}{15} \times \frac{9}{14} + \frac{6}{16} \times \frac{10}{15} \times \frac{9}{14} = 0.482$

c) $P(2 \text{ girls and } 1 \text{ boy}) = \frac{10}{16} \times \frac{6}{15} \times \frac{5}{14} + \frac{6}{16} \times \frac{10}{15} \times \frac{5}{14} + \frac{6}{16} \times \frac{5}{15} \times \frac{10}{14} = 0.268$

d) $P(3\,\text{girls}) = \frac{6}{16} \times \frac{5}{15} \times \frac{4}{14} = 0.0357$

3. a) $\frac{1}{21}$ b) $\frac{10}{21}$ c) $\frac{11}{21}$

4. a) $\frac{1}{816}$ b) $\frac{7}{102}$ c) $\frac{7}{34}$

5. $P(engineer \mid mangerial) = \dfrac{0.2 \times 0.075}{0.2 \times 0.075 + 0.2 \times 0.5 + 0.6 \times 0.2} = 0.405$

6. $P(\text{Noaccident / Triggeredal arm}) = \dfrac{0.9 \times 0.02}{0.1 \times 0.097 + 0.02 \times 0.9} = 0.157$

## Lesson 6: Probability distributions: Discrete probability distributions

### a) Learning objective:

Appreciate the application of probability ( distribution) in production, finance, and economics

### b) Teaching resources:

*Manila papers, calculators, coins, dice, etc.*

### c) Prerequisites/Revision/Introduction:

Students will perform well in this lessons if they try to relate the content for all previous lessons for this units to real life problems involving events of chance and also if there are enough skilled on tree and Venn diagrams.

### d) Learning activities

- Invite students to work in small groups, attempt the learning activity 5.2.1. and answer to related questions;
- Ask neighbouring groups to share their answers and support each other by exchanging arguments on that activity;
- Invite group representatives to resent the findings in a whole class discussions;
- Teacher harmonizes answers for students on the learning **activity 5.2.1.**
- Guide students to discuss and apply discrete **probability distributions** to compute expectation, variance and standard deviation.
- Guide student to discuss and apply **discrete probability distributions** to determine probability.

- Use different probing questions and guide them to explore **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- Let them know that for each **distribution,** the expectation, variance and standard deviation can be computed.
- Use different probing questions and guide students to explore different **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- After this step, guide students to do the **application activity 5.2.1.**, and evaluate whether lesson objectives were achieved.
- After this step, guide Students to go through the **application activity 5.2.1.,** and evaluate whether lesson objectives were achieved.

**Answer of learning activity 5.2.1.**

| X | Frequency | f(x)=p(x) |
|---|---|---|
| 0 | 1,218 | 1,218/101,501=0.012 |
| 1 | 32,379 | 32,379/101,501=0.319 |
| 2 | 37,961 | 37,961/101,501=0.374 |
| 3 | 19,387 | 19,387/101,501=0.191 |
| 4 | 7,714 | 7,714/101,501=0.076 |
| 5 | 2,842 | 2,842/101,501=0.028 |
| Total | 101,501 | |

The required conditions for a discrete probability function are:

1. $f(x) \geq 0$
2. $\sum f(x) = 1$

**e) Answer of Application activity 5.2.1.**

The return on bond X is $5\,000\,000 Frw \times 4\% = 200\,000 Frw$. The expected return then is $E(X) = 200\,000 \times (0.98) Frw - 5000\,000(0.02) Frw = 96\,000\, frw$

The return on bond Y is $5\,000\,000(2^{\frac{1}{2}}\%)\,frw = 125\,000\,frw$ . The expected return then is $E(X) = 125\,000(0.99)\,frw - 5\,000\,000(0.01)\,frw = 73\,750\,frw$

Hence, bond X would be a better investment since the expected return is higher

## Lesson 7: Probability distributions: Binomial distribution

### a) Learning objective:

Appreciate the application of probability ( distribution) in production, finance, and economics

### b) Teaching resources:

*Manila papers, calculators, coins, dice, etc.*

### c) Prerequisites/Revision/Introduction:

Students will perform well in this lessons if they try to relate the content for all previous lessons for this units to real life problems involving events of chance and also if there are enough skilled on tree and Venn diagrams.

### d) Learning activities

- Invite students to work in small groups, attempt the learning activity 5.2.2. and answer to related questions;
- Ask neighbouring groups to share their answers and support each other by exchanging arguments on that activity;
- Invite group representatives to resent the findings in a whole class discussions;
- As a Teacher harmonizes answers for students on the learning **activity 5.2.2.**
- Guide students to discuss and apply binomial **probability distributions** to compute expectation, variance and standard deviation.
- Guide student to discuss and apply binomial experiment to determine probability.
- Use different probing questions and guide them to explore **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- Let them know that for each **binomial distribution,** the expectation, variance and standard deviation can be computed.

- Use different probing questions and guide students to explore different **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- After this step, guide students to do the **application activity 5.2.2.,** and evaluate whether lesson objectives were achieved.
- After this step, guide students to go through the **application activity 5.2.2.,** and evaluate whether lesson objectives were achieved.

**Answer of learning activity 5.2.2.**

We can think of many examples of random trials or experiments in which there are two basic outcomes of a qualitative nature:

- the coin comes up either heads or tails,
- the part coming off the assembly line is either defective or not defective,
- it either rains today or it doesn't, and so forth

**e) A nswer of application activity 5.2.2**

1. A multiple-choice test with 20 questions has five possible answers for each question. A completely unprepared student picks the answers for each question at random and independently. Suppose X is the number of questions that the student answers correctly. Calculate the probability that:

a) The student gets every answer wrong

$$Given \quad that \quad n=20, p=\frac{1}{5}, q=\frac{4}{5} \Rightarrow a) P(X=0)={}^{20}C_0 p^0 q^{20}=\frac{20!}{0!20!}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{20}=0.01153$$

b) The student gets every answer right.

$$Given \quad that \quad n=20, p=\frac{1}{5}, q=\frac{4}{5} \Rightarrow b) \quad P(X=20)={}^{20}C_{20} p^{20} q^0=\frac{20!}{20!0!}\left(\frac{1}{5}\right)^{20}\left(\frac{4}{5}\right)^0=1.05\times 10^{-14}$$

c) The student gets 8 right answers.

$$Given \quad that \quad n=20, p=\frac{1}{5}, q=\frac{4}{5} \Rightarrow c) \quad P(X=8)={}^{20}C_8 p^8 q^{12}=\frac{20!}{8!12!}\left(\frac{1}{5}\right)^8\left(\frac{4}{5}\right)^{12}=0.02216$$

2. An examination consisting of **10 multiple choice** questions is to be done by a candidate who has not revised for the exam. Each question has five possible answers out of which only one is correct. The candidate simply decides to guess the answers.

**(i)** What is the probability that the candidate gets no answer correct?

$$Given \quad that \quad n=10, p=\frac{1}{5}, q=\frac{4}{5} \Rightarrow a) \quad P(X=0)={}^{10}C_0 p^0 q^{10}=\frac{10!}{0!10!}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{10}=0.1073741824$$

iii) What is the probability that the student gets two correct answers?

$$Given \quad that\, n=10, p=\frac{1}{5}, q=\frac{4}{5} \Rightarrow b) P(X=2)={}^{10}C_2 p^2 q^2=\frac{10!}{2!8!}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^8=0.30199 \cong 0.302$$

There are 45 ways to get exactly two correct answers, and that each such outcome has probability 0.006712. Multiplying the two produces a probability of 0.302. therefore, the probability that the candidate gets two answers correct is 0.302.

**Answer:**

| Outcomes | | | Probability of Outcome F(X) |
|---|---|---|---|
| 1st coin | 2nd coin | 3rd coin | |
| H | H | H | (0.5)(0.5)(0.5) = 0.125 |
| H | H | T | (0.5)(0.5)(0.5) = 0.125 |
| H | T | H | (0.5)(0.5)(0.5) = 0.125 |
| T | H | H | (0.5)(0.5)(0.5) = 0.125 |
| H | T | T | (0.5)(0.5)(0.5) = 0.125 |
| T | H | T | (0.5)(0.5)(0.5) = 0.125 |
| T | T | H | (0.5)(0.5)(0.5) = 0.125 |
| T | T | T | (0.5)(0.5)(0.5) = 0.125 |
| | | | $\sum f(X) = 1.00$ |

Thus there are 8 possible outcomes.

This shows a theoretical probability distribution. Each of the 8 outcomes is equally likely.

i) Thus the probability for any one of them is = 0.125.

ii) There is only one outcome in which 3Heads appear: Thus P (3H) = 1/8 = 0.125

iii) There are three outcomes in which two heads appear. Thus P(2H) = = 0.375

## Lesson 8: Probability distributions: Bernoulli distribution

**a) Learning objective:**

Appreciate the application of probability ( distribution) in production, finance, and economics

**b) Teaching resources:**

*Manila papers, calculators, coins, dice, etc.*

**c) Prerequisites/Revision/Introduction:**

Students will perform well in this lesson if they try to relate the content for all previous lessons for this unit to real life problems involving events of chance and also if there are enough skilled basic concepts especially on tree and Venn diagrams.

**d) Learning activities**

- Invite students to work in small groups, attempt the learning activity 4.2.3. and answer to related questions;
- Ask neighbouring groups to share their answers and support each other by exchanging arguments on that activity;
- Invite group representatives to resent the findings in a whole class discussions;
- As a Teacher harmonizes answers for students on the learning **activity 5.2.3.**
- Guide students to discuss and apply Bernoulli distribution to compute expectation, variance and standard deviation.
- Guide student to discuss and apply Bernoulli distribution to determine probability.
- Use different probing questions and guide them to explore **examples**

given in the student's book and lead them to realize that probability is applicable in the real life.

- Let them know that for each **Bernoulli distribution,** the expectation, variance and standard deviation can be computed.
- Use different probing questions and guide students to explore different **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- After this step, guide students to do the **application activity 5.2.3.,** and evaluate whether lesson objectives were achieved.
- After this step, guide students to go through the **application activity 5.2.3.,** and evaluate whether lesson objectives were achieved.

**Answer of learning activity 5.2.3.**

The Bernoulli examples (tossing coin once, answering question which has true or false, etc) , using the binomial experiment shown above (voting, planning for baby, passing exam with multiple choice question, etc) differences (binomial occurs more times while Bernoulli is done once) and Similarities (both have two outcomes)

### e) Answers of application activity 5.2.3

**Answers:** tossing a coin once: The only two possible outcomes are heads and tails., answering a multiple choice question: The possible outcomes are Yes/ True and No/False., Rolling a die where a '1' is a 'success', all other numbers are considered as 'failure', voting one candidate from two candidates. If, the new born child is a girl or boy?

## Lesson 9: Probability distributions: Poisson distribution

### a) Learning objective:

Appreciate the application of probability ( distribution) in production, finance, and economics

### b) Teaching resources:

*Manila papers, calculators, coins, dice, etc.*

### c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they try to relate the content for all previous lessons for this unit to real life problems involving events of chance and also if there are enough skilled basic concepts especially on tree and Venn diagrams.

**d) Learning activities**

- Invite students to work in small groups, attempt the learning activity 5.2.4. and answer to related questions;
- Ask neighbouring groups to share their answers and support each other by exchanging arguments on that activity;
- Invite group representatives to resent the findings in a whole class discussions;
- As a Teacher harmonizes answers for students on the learning **activity 5.2.4.**
- Guide students to discuss and apply Poisson distribution to compute expectation, variance and standard deviation.
- Guide student to discuss and apply Poisson distribution_to determine probability.
- Use different probing questions and guide them to explore **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- Let them know that for each Poisson distribution**,** the expectation, variance and standard deviation can be computed.
- Use different probing questions and guide students to explore different **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- After this step, guide students to do the **application activity 5.2.4.,** and evaluate whether lesson objectives were achieved.
- After this step, guide students to go through the **application activity 5.2.4.,** and evaluate whether lesson objectives were achieved.

**Answer of learning activity 5.2.4.**

a) The number of occurrences/successes that occur in any interval is independent of the number of occurrences.
b) The probability of occurrences /successes in an interval is the same for all equal Size intervals
c) The probability of occurrences /success in an interval is proportional to the size of the interval.
d) The probability of more than one occurrences/ success in an interval

approaches 0 as the interval becomes smaller.

e) The probability of an occurrence /success of the event is the same for any two Intervals of equal length.

f. These skills are needed in accounting professional because accountant need to know the number of occurrences/success of any activity that occur in any interval and the probability of that occurrences/success in an interval or of the event

### e) Answer of application activity 5.2.4

1. We want to determine that the probability that a Poisson random variable with a mean of 1.5 is equal to 0. We have $\mu = 1.5$; $x = 0$ and we substitute these values in the formula $P(X) = \frac{\mu^x e^{-\mu}}{x!} = \frac{(1.5)^0 e^{-1.5}}{1} = e^{-1.5} = 0.2231$

   Thus, the probability that in 100pages selected there are no typing error is 0.2231

2. First, find the mean number l of errors. Since there are 200 errors distributed over 500 pages, each page has an average of $\lambda = \frac{200}{500} = \frac{2}{5} = 0.4$

   Or 0.4 error per page. Since $X = 3$, substituting into the formula gives

   $$P(X,\lambda) = \frac{e^{-\lambda}\lambda^x}{3!} = \frac{(2.7183)^{-0.4}(0.4)^3}{3!} = 0.0072.$$

   Thus, there is less than a 1% chance that any given page will contain exactly 3 errors

**Solution:**

$X \sim P_0(4)$

$P(X = x) = \frac{e^{-4}4^x}{x!}$

Where X is the random variable. “the number of bacteria in 1 ml of liquid”

- $P(X = 0) = P(no\ \ bacteria) = e^{-4} = 0.0183$
- $P(X = 4) = P(4\ bacteria\ in\ 1\ ml) = \frac{e^{-4}4^4}{4!} = 0.195$
- $P(X < 3) = P(less\ \ than\ 3\,bacteria\ in\ 1\ ml)$

  $= P(X = 0) + P(X = 1) + P(X = 2)$

  $= e^{-4}(1 + 4 + 8) = 0.238$

## Lesson 10: Probability distributions: Continuous random distributions

### a) Learning objective:

Appreciate the application of probability ( distribution) in production, finance, and economics

### b) Teaching resources:

*Manila papers, calculators, coins, dice, etc.*

### c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they try to relate the content for all previous lessons for this unit to real life problems involving events of chance and also if there are enough skilled basic concepts especially on tree and Venn diagrams.

### d) Learning activities

- Invite students to work in small groups, attempt the learning activity 5.2.5. and answer to related questions;
- Ask neighbouring groups to share their answers and support each other by exchanging arguments on that activity;

- Invite group representatives to resent the findings in a whole class discussions;
- As a Teacher harmonizes answers for students on the learning **activity 5.2.5.**
- Guide students to discuss and apply Continuous random distributions to compute expectation, variance and standard deviation.
- Guide student to discuss and apply Continuous random distributionsto determine probability.
- Use different probing questions and guide them to explore **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- Let them know that for each Poisson distribution**,** the expectation, variance and standard deviation can be computed.
- Use different probing questions and guide students to explore different **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- After this step, guide students to do the **application activity 5.2.5.,** and evaluate whether lesson objectives were achieved.
- After this step, guide students to go through the **application activity 5.2.5.,** and evaluate whether lesson objectives were achieved.

## Answer of learning activity 5.2.5

1. Observations from students are vary according to their own observations. As teacher harmonizes their answers basing on continuous probability distributions.

2. from the given function above:

$$E(X)=\int_0^1 x\,f(x)dx=\int_0^1 x(\frac{3}{4}(1+x^2))dx=\frac{9}{16},\quad \text{details are here by}$$

$$E(X)=\int_{all\ x} f(x)\,dx=\frac{3}{4}\int_0^1 x\left(1+x^2\right)dx=\frac{3}{4}\int_0^1\left(x+x^3\right)dx=\frac{3}{4}\left[\frac{x^2}{2}+\frac{x^4}{4}\right]_0^1=\frac{3}{4}\left(\frac{1}{2}+\frac{1}{4}\right)$$

$$=\frac{3}{4}\left(\frac{3}{4}\right)=\frac{9}{16}=0.5625$$

$$But\ E\left(X^2\right)=\int_{all\,x} x^2 f(x)\,dx=\frac{3}{4}\int_0^1\left(x^2+x^4\right)dx=\frac{3}{4}\left[\frac{x^3}{3}+\frac{x^5}{5}\right]=\frac{3}{4}\left[\frac{1}{3}+\frac{1}{5}\right]=\frac{3}{4}*\frac{8}{15}=\frac{2}{5}=0.4$$

$$Var\left(X\right)=\int_{all\,x} x^2 f\left(x\right)dx-\mu^2$$

$$Then,\ Var\ X=0.4-\left(0.5625\right)^2=0.0835$$

$$\sigma_X=\sqrt{Var\ X}=0.289$$

## e) Answers of application activity 5.2.5.

Find the constant c such that the function $f(x)=\begin{cases} cx^2, & 0<x<3 \\ 0 & \text{otherwise} \end{cases}$ is a density function.

$$\int_0^3 cx^2dx = 1 \Leftrightarrow c\left(\frac{x^3}{3}\right)_0^3 = 1 \Leftrightarrow 9c = 1 \Rightarrow c = \frac{1}{9}$$

a) Compute $P(1<X<2)=\int_1^2 cx^2dx=\int_1^2 \frac{1}{9}x^2dx=\frac{7}{27}$

b) How can you compute $P(-\infty<X<+\infty)$? this give us $\int_{-\infty}^{+\infty}\frac{1}{9}x^2dx=0$

c) Results in a varies as interval varies but b remain 0.

## Lesson 10: Probability distributions: Normal distributions

### a) Learning objective:

Appreciate the application of probability ( distribution) in production, finance, and economics

### b) Teaching resources:

*Manila papers, calculators, coins, dice, etc.*

### c) Prerequisites/Revision/Introduction:

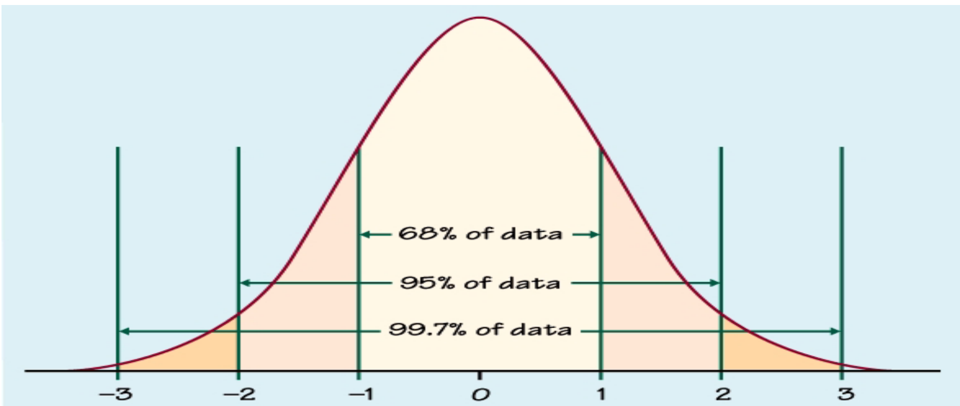
Students will perform well in this lesson if they try to relate the content for all previous lessons for this unit to real life problems involving events of chance and also if there are enough skilled basic concepts especially on tree and Venn diagrams.

### d) Learning activities

- Invite students to work in small groups, attempt the learning activity 5.2.6. and answer to related questions;
- Ask neighbouring groups to share their answers and support each other by exchanging arguments on that activity;

- Invite group representatives to resent the findings in a whole class discussions;
- As a Teacher harmonizes answers for students on the learning **activity 5.2.6.**
- Guide students to discuss and apply Normal distributions to compute expectation, variance and standard deviation.
- Guide student to discuss and apply  normal distributionsto determine probability.
- Use different probing questions and guide them to explore **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- Let them know that for each Poisson distribution**,** the expectation, variance and standard deviation can be computed.
- Use different probing questions and guide students to explore different **examples** given in the student's book and lead them to realize that probability is applicable in the real life.
- After this step, guide students to do the **application activity 5.2.6.,** and evaluate whether lesson objectives were achieved.
- After this step, guide students to go through the **application activity 5.2.6.,** and evaluate whether lesson objectives were achieved.

**Answer of learning activity 5.2.6.**




Yes, there is a linkage because:

1. **Approximately 68%** of the observations fall within 1 standard deviation of the mean,
2. **Approximately 95%** of the observations fall within 2 standard deviations of the mean,

3. And virtually all the observations (**approximately 99.7%**) fall within 3 standard deviations of the mean.

**Note:** All of these percentages represent the likelihood that the dataset will be covered during research-related tasks. These findings, in descriptive statistics, show the degree of symmetry, whereas in probability, they show the extent to which the data set is covered.

**e) Application activity 5.2.6.**

1. $P(-1 < Z < 3) = \Phi(-1) + \Phi(3) = 0.84$
   $P(-3 < Z < 3) = \Phi(-3) + \Phi(3) = 2\Phi(3) = 0.9973$

**Notice:** One of the normal distribution full tables is attached on an appendix to the students' book, but the teacher assist them in reading and using it.

2. To solve the problem, find the area under a normal distribution curve to the left of 15.

**Step 1**

Draw a figure and represent the area as shown in the following figure.


**Step 2** Find the z value for 15. $z = \frac{X - \mu}{\sigma} = \frac{15 - 25}{4.5} = -2.22$

**Step 3** Find the area to the left of $z = -2.22$. It is 0.0132.

Step 4 To find how many calls will be made in less than 15 minutes, multiply the sample size 80 by 0.0132 to get 1.056. Hence, 1.056, or approximately 1, call will be responded to in under 15 minutes.

## Lesson 11: Application of Probability and probability distributions in business, finance, economics, and production related issues.

### a) Learning objective:

Appreciate the application of probability in production, finance, and economics.

### b) Teaching resources:

*Manila papers, calculators, coins, dice, etc.*

### c) Prerequisites/Revision/Introduction:

- Students will perform well in this lesson if they try to relate the content for all previous lessons for this unit to real life problems involving events of chance and also if there are enough skilled basic concepts especially on tree and Venn diagrams.

### d) Learning activities

- Invite students to work in small groups, attempt the learning activity 5.3. and answer to related questions;
- Ask neighbouring groups to share their answers and support each other by exchanging arguments on that activity;
- Invite group representatives to resent the findings in a whole class discussions;
- As a Teacher harmonizes answers for students on the learning **activity 5.3.**
- Guide students to discuss and apply probability distribution to compute expectation, variance and standard deviation.
- Guide student to discuss and apply probability and probability distributions to handle the applications.
- Use different probing questions and guide them to explore **examples** given in the student's book and lead them to realize that probability and probability distributions are applicable in the real life.
- Let them know that for each Poisson distribution**,** the expectation, variance and standard deviation can be computed.
- Use different probing questions and guide students to explore different **examples** given in the student's book and lead them to realize that probabilities are applicable in the real life.

- After this step, guide students to do the **application activity 5.3.,** and evaluate whether lesson objectives were achieved.

## Answer of learning activity 5.3.

**a)** $P(5)=\dfrac{56}{127}$.

**b)** $P(\textit{fewer than } 6 \textit{ days})=\dfrac{15}{127}+\dfrac{32}{127}+\dfrac{56}{127}=\dfrac{103}{127}=0.8$.

(Fewer than 6 days means 3, 4, or 5 days)

**c)** $P(\textit{at most } 4 \textit{ days})=\dfrac{15}{127}+\dfrac{32}{127}=\dfrac{47}{127}=0.37$.

(At most 4 days means 3 or 4 days)

**d)** $P(\textit{at least } 5 \textit{ days})=\dfrac{56}{127}+\dfrac{19}{127}+\dfrac{5}{127}=\dfrac{80}{127}=0.6$.

(At least 5 days means 5,6, or 7 days)

## e) Answer of application activity 5.3

1. P(A or C) = P(A)+P(B) = 0.025+0.075=0.10
2. Payoff Table:

| | **Demand** | | | |
|---|---|---|---|---|
| **Produce** | $Sell_o$ | $Sell_1$ | $Sell_2$ | $Sell_3$ |
| $Bake_o$ | 0 | 0 | 0 | 0 |
| $Bake_1$ | -3.00 | 5.00 | 5.00 | 5.00 |
| $Bake_2$ | -6.00 | 2.00 | 10.00 | 10.00 |
| $Bake_3$ | -9.00 | -1.00 | 7.00 | 15.00 |

b) Assuming probability of each event (bake cakes) is equal: Sell =0 .25

EMV(0) = \$ 0

EMV(1) =0 .25(-3) +0 .25(5) + 0.25(5) + 0.25(5)= \$3.00

EMV(2) = 0.25(-6) +0 .25(2) +0.25(10) +0 .25(10)= \$4.00

EMV(3) =0 .25(-9) +0 .25(-1) +0.25(7) + 0.25(15)= \$3.00

EMV* decision is to bake 2 cakes.

EVPI = EPC – EMV

EPC=0 .25(-3) +0 .25(5) +0 .25(10) +0 .25(15)= \$6.75

EMV = \$4.00

EVPI = \$6.75 – 4.00 = \$2.75

## 5.6. Unit summary

**Probability of an event**

The probability of an event $A \subset \Omega$, is a real number obtained by applying to $A$ the function $P$ defined by

$$P(A) = \frac{Number\ of\ outcomes\ in\ A}{Total\ number\ of\ outcomes\ in\ the\ sample\ space} = \frac{n(A)}{n(\Omega)}$$

When $E$ and $E'$ are complementary events, then $P(E) = 1 - P(E')$.

When two events A and B are not mutually exclusive, $A \cap B = \phi$ the probability that A or B occurs is given by: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A **probability distribution** is a listing of all possible outcomes of an experiment and the probability associated with each outcome**.** Probability distributions can be divided into: **discrete probability distributions, and continuous probability distributions.**

**Discrete Random variable:** Within a range of numbers, discrete random variables can take on only a finite number of values. Therefore, if you toss a coin the number of heads is a discrete random variable. Also, a variable is said to be random if it takes on different values as the result of the outcomes of a random experiment.

In probability distribution the random variable is said to be **discrete** if it is only allowed to take on **countable values.**

**A continuous random variable** is a random variable that can theoretically assume any value between two given variables.

For a continuous random variable, there are an infinite number of values in an interval and for that matter, we cannot talk of the probability that the random variable will take on a specific value but instead **we talk of the probability that a continuous random variable will be within a specific interval.**

For a continuous random variable, f(x) provides the value of the function at any particular value of x.

It does not provide directly the probability of the random variable taking on some specific value.

The area under the graph of f(x), corresponding to some interval, however, provides the probability that the continuous random variable will take on a value in that interval.

**The total area under a continuous curve representing the probability distribution is equal to 1.**

If you let X as a continuous random variable. Then a probability distribution or probability density function (pdf) of X is a function f(x) such that for any two numbers a and b with a ≤ b, we have $P(a \le X \le b) = \int_a^b f(x)dx$ .

## 5.7. End unit assessment

### Answers for End unit assessment

1. Choose a letter at random from the word SCHOOL

2. $\frac{9}{20}$

3. $\frac{21}{46}$

4. $\frac{3}{13}$

5. a. $\frac{1}{6}$ b. $\frac{5}{126}$

6. 0.13

7. 0.56

8. a) 0.34 b) 0.714 c) 0.0833

9. a) 0.384 b) 0.512

10. $\frac{3}{13}$

11.

a) $P(A)=P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)$

b) $P(B_1|A)=\dfrac{P(B_1\cap A)}{P(A)}=\dfrac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$

$$P(B_2|A)=\frac{P(B_2\cap A)}{P(A)}=\frac{P(A|B_2)P(B_2)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

$$P(B_3|A)=\frac{P(B_3\cap A)}{P(A)}=\frac{P(A|B_3)P(B_3)}{P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)}$$

**Generally,**

$$P(B_i|A)=\frac{P(B_i\cap A)}{P(A)}=\frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{3}P(A|B_i)P(B_i)}$$

## 5.8 Additional activities

### 5.8.1 Remedial activity

1. A card is drawn at random from an ordinary pack of 52 playing cards. Find the probability that the card drawn is a club or a diamond. A die is thrown once.

**Solution:**

There are 13 clubs, then $P(\text{club}) = \frac{13}{52}$

There are 13 diamonds, then $P(\text{diamond}) = \frac{13}{52}$

Since a card cannot be both a club and a diamond, $P(\text{club} \cap \text{diamond}) = 0$

Therefore, P(a club or a diamond) = P(club) + P(diamond) $\frac{13}{52} + \frac{13}{52} = \frac{26}{52} = \frac{1}{2}$

2. A pen is drawn from a basket containing 10 pens of which 5 are red and 3 are black. If A is the event: "a pen is red" and B is the event: "a pen is black", find $P(A), P(B), P(A \cup B)$.

**Solution**

There are 5 red pens, then $P(A) = \frac{5}{10} = \frac{1}{2}$

There are 3 black pens, then $P(B) = \frac{3}{10}$

Since the pen cannot be red and black at the same time, then $A \cap B = \varnothing$ and two events are mutually exclusive so $P(A \cup B) = \frac{1}{2} + \frac{3}{10} = \frac{8}{10}$

### 5.8.2. Consolidation activity

**1)** In a group of 20 adults, 4 out of the 7 women and 2 out of the 13 men wear glasses. What is the probability that a person chosen at random from the group is a woman or someone who wears glasses?

**Solution**

Let $A$ be the event: "the person chosen is a woman".

$B$ be the event: "the person chosen wears glasses".

Now, there are 7 women, then $P(A)=\frac{7}{20}$

There are 6 persons who wear glasses, then $P(B)=\frac{6}{20}$

There are 4 women who wear glasses, then $P(A\cap B)=\frac{4}{20}$

The probability that a person chosen at random from the group is a woman or someone who wears glasses is given by $P(A\ or\ B)$ which is

$$
\begin{aligned}
P(A\cup B) &= P(A)+P(B)-P(A\cap B)\\
&=\frac{7}{20}+\frac{6}{20}-\frac{4}{20}\\
&=\frac{9}{20}
\end{aligned}
$$

**On the other hand:** There are 7 women and 6 persons who wear glasses but also 4 women who wear glasses. Then, $A\cup B=9$ and $P(A\cup B)=\frac{9}{20}$.

3. A coin is weighted so that heads is three times as likely to appear as tails. Find $P(H)$ and $P(T)$.

**Solution**

Let $P(T)=p_1$, then $P(H)=3p_1$.

But $P(H)+P(T)=1$

Therefore $p_1+3p_1=1\Leftrightarrow 4p_1=1\Rightarrow p_1=\frac{1}{4}$

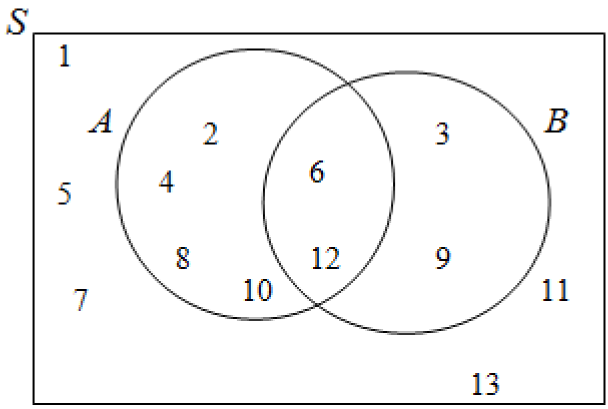
Thus, $P(H)=\frac{3}{4}$ and $P(T)=\frac{1}{4}$.

### 5.8.3 Extended activity

1. An integer is chosen at random from the set $S=\{x: x\in\mathbb{Z}^{+}, x<14\}$. Let A be the event of choosing a multiple of 2 and let B be the event of choosing a multiple of 3.

Find $P(A\cup B)$, $P(A\cap B)$ and $P(A-B)$.

**Solution**




From the diagram, $\#S=13$

$A\cup B=\{2,3,4,6,8,9,10,12\}\Rightarrow \#(A\cup B)=8,$ thus $P(A\cup B)=\frac{8}{13}$

$A\cap B=\{6,12\}\Rightarrow \#(A\cap B)=2,$ thus $P(A\cap B)=\frac{2}{13}$

$A-B=\{2,4,8,10\}\Rightarrow \#(A-B)=4,$ thus $P(A-B)=\frac{4}{13}$

**Solution**

1. In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

   a) A person has type O blood.

   b) A person has type A or type B blood.

   c) A person has neither type A nor type O blood.

   d) A person does not have type AB blood.

**Solution**

| Type | Frequency |
|---|---|
| A | 22 |
| B | 5 |
| AB | 2 |
| O | 21 |
| Total | 50 |

They are mutually exclusive.

a) $P(O)=\frac{f}{n}=\frac{21}{50}$

b) $P(\text{A or B})=\frac{22}{50}+\frac{5}{50}=\frac{27}{50}$

c) Neither A nor O means that a person has either type B or type AB blood.)

$$P(\text{neither A nor O})=\frac{5}{50}+\frac{2}{50}=\frac{7}{50}.$$

d) Find the probability of not AB by subtracting the probability of type AB

from 1.

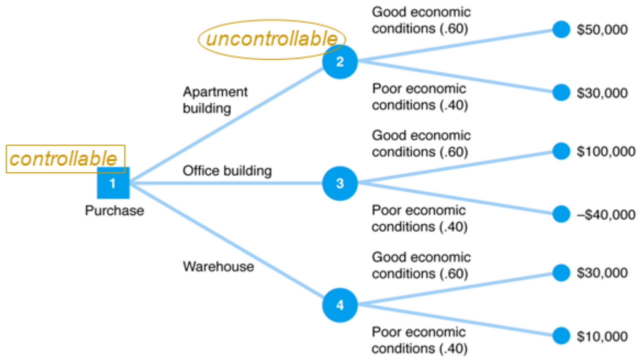
$$P(\text{not AB})=1-P(AB)=1-\frac{2}{50}=\frac{48}{50}=\frac{24}{25}$$

**Additional examples**

1. Using tree diagram in the following payoff table for Real Estate Investment, you can easily choose the optimal investment (best alternative among others).

| | States of Nature | |
|---|---|---|
| Decision (Purchase) | Good Economic Conditions (0.60) | Poor Conditions (0.40) |
| Apartment Building | \$50,000 | \$30,000 |
| Office building | 100,000 | -40,000 |
| Werehouse | 30,000 | 10,000 |

- All these information can be represented on the tree diagram as follows:




- The expected value is computed *at each probability (uncontrollable) node*:

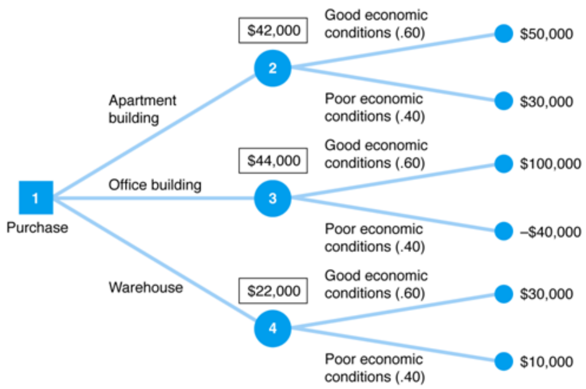
**EV**(node 2) = .60(\$50,000) + .40(30,000) = \$42,000

**EV**(node 3) = .60(\$100,000) + .40(-40,000) = \$44,000

**EV**(node 4) = .60(\$30,000) + .40(10,000) = \$22,000

Populating the decision tree from right to left.

- The branch (es) with the greatest expected value are then selected, starting from the left and progressing to the right.



The decision maker will select the alternative with the largest expected payoff if maximizing expected payoff is the decision criterion. In this case, “office building” is the best.

2. Mr. X earned a score of 940 on a national achievement test. The mean test score was 850 with a standard deviation of 100. What proportion of students had a higher score than Mr. X? (Assume that test scores are normally distributed.)

(A) 0.10

(B) 0.18

(C) 0.50

(D) 0.82

(E) 0.90

**Answer**

The correct answer is B. As part of the solution to this problem, we assume that test scores are normally distributed. In this way, we use the normal distribution as a model for measurement. Given an assumption of normality, the solution involves three steps.

- First, we transform Mr. X’s test score into a z-score, using the z-score transformation equation. $z = (X - \mu) / \sigma = (940 - 850) / 100 = 0.90$
- Then from the standard normal distribution table, we find the cumulative probability associated with the z-score. In this case, we find $P(Z < 0.90) = 0.8159$.
- Therefore, the $P(Z > 0.90) = 1 – P(Z < 0.90) = 1 - 0.8159 = 0.1841$. Thus, we estimate that 18.41 percent of the students tested had a higher score than Mr. X

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